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On properties related to star countability [☆]

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ABSTRACT

We prove that a Hausdorff metaLindelöf weakly star countable space is feebly Lindelöf and a Hausdorff metacompact weakly star finite space is almost compact which partially answers a question of Alas and Wilson (2017) [2, Question 3.14]. We also obtain a normal example of a weakly star countable space which is neither almost star countable nor star Lindelöf without any set-theoretic assumptions, which answers a question implicitly asked by Song (2015) [13, Remark 2.8] and a question asked by Alas, Junqueira and Wilson (2011) [3, Question 4]. Under MA+¬CH, there even exists a normal weakly star countable Moore space which is not almost star countable. An example of a Tychonoff star compact and weakly star finite space which is not star countable is also given. Finally, we prove that every weakly star countable Hausdorff space with a rank 4-diagonal has cardinality at most 2^ω.

1. Introduction

All topological spaces in this paper are assumed to be Hausdorff unless otherwise stated.

If A is a subset of a space X and $\mathcal{U}$ is a family of subsets of X, then the *star* of a subset $A \subset X$ with respect to $\mathcal{U}$ is the set

$$\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

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We also put $\mathrm{St}^0(A,\mathcal{U}) = A$ and for a natural number n, $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^{n}(A,\mathcal{U}),\mathcal{U})$. As usual, we write $\mathrm{St}^{n}(x,\mathcal{U})$ instead of $\mathrm{St}^{n}(\{x\},\mathcal{U})$ for simplicity.

Let P be a topological property. A space X is said to be *star* P, if for any open cover $\mathcal{U}$ of X there is a subset $A \subset X$ with property P such that $\mathrm{St}(A,\mathcal{U}) = X$. The space X is *weakly star* P (respectively, *almost star* P) if given any open cover $\mathcal{U}$ of X, there is a subset $A \subset X$ with property P such that $\mathrm{cl}_X(\mathrm{St}(A,\mathcal{U})) = X$ (respectively, $\bigcup\{\mathrm{cl}_X(\mathrm{St}(x,\mathcal{U})) : x \in A\} = X$).

The term "star P" was defined by van Mill, Tkachuk and Wilson in [19], but many star properties, such as "star finite" and "star countable" were introduced by Ikenaga (see for example [7]) and first studied by van Douwen, Reed, Roscoe and Tree in [17] and later by many other authors. A survey of star covering properties with a comprehensive bibliography can be found in [10]. The terms "weakly star countable" and "almost star countable" were first used and studied by Song and Xuan in (2018) [12] and (2015) [13] respectively. Alas and Wilson in (2017) [2] later studied the relationship between the properties of being weakly star countable, that of being weakly star finite and feeble compactness, and obtained many interesting results concerning these properties. We should remind the reader that "weakly star finite" and "weakly star countable" in [10] are also called "1-cl-starcompact" (also "1-H-closed") and "1-cl-star-Lindelöf" respectively.

In this paper we continue to investigate the properties of being almost star countable and of being weakly star countable. In particular, we prove that a Hausdorff metaLindelöf weakly star countable space is feebly Lindelöf (Theorem 3.2) and a Hausdorff metacompact weakly star finite space is almost compact (and hence, feebly compact) (Theorem 3.3), which gives a partial answer to a question of Alas and Wilson (2017) [2, Question 3.14]. We also obtain a normal example of a weakly star countable space which is neither almost star countable nor star Lindelöf without any set-theoretic assumptions (Example 3.7), which answers a question implicitly asked by Song (2015) [13, Remark 2.8] and a question asked by Alas, Junqueira and Wilson (2011) [3, Question 4]. Under MA+¬CH, there even exists a normal weakly star countable Moore space which is not almost star countable (Example 3.8). An example of a Tychonoff star compact and weakly star finite space which is not star countable is also given (Example 3.11). Finally, we prove that every weakly star countable Hausdorff space with a rank 4-diagonal has cardinality at most 2^{ω} (Theorem 3.17).

2. Notation and terminology

The cardinality of a set X is denoted by $|X|$, and $[X]^2$ will denote the set of two-element subsets of X.

Definition 2.1. A space X is *metaLindelöf* if every open cover of X has a point-countable open refinement.

Definition 2.2. A space X is *metacompact* if every open cover of X has a point-finite open refinement.

Definition 2.3. A space X is *almost compact* (or *H-closed*) if for every open cover $\mathcal{U}$ of X there exists a finite subfamily of $\mathcal{U}$ whose union is dense in X.

Definition 2.4. A space X has the *countable chain condition* (CCC for short) if any disjoint family of non-empty open sets in X is countable.

Definition 2.5. A space X has the *discrete countable chain condition* (DCCC for short) if any discrete family of non-empty open sets in X is countable.

Definition 2.6. The *extent* of a space X, denoted by $e(X)$, is the supremum of the cardinalities of closed and discrete subsets of X.