Topology and its Applications 247 (2018) 81–90

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On dynamics generated by a uniformly convergent sequence of maps

Puneet Sharma *, Manish Raghav

Department of Mathematics, I.I.T. Jodhpur, N.H. 65, Nagaur Road, Karwar, Jodhpur 342037, India

ARTICLE INFO

Article history:
Received 20 March 2017
Received in revised form 26 July 2018
Accepted 26 July 2018
Available online 30 July 2018

MSC:
37B20
37B55
54H20

Keywords:
Non-autonomous dynamical systems
Equicontinuity
Proximal pairs
Various notions of mixing and sensitivities

ABSTRACT

In this paper, we investigate the dynamics of a non-autonomous dynamical system $(X, \mathbb{F})$ generated by a uniformly convergent sequence of continuous self maps on X. We relate the dynamical behavior of $(X, \mathbb{F})$ with the dynamical behavior of the limiting system (and vice versa). In the process, we relate properties like equicontinuity, minimality, various forms of mixing and sensitivities, dense periodicity and denseness of proximal pairs (cells) for the two systems. We also give examples to investigate the necessity of the conditions imposed.

1. Introduction

Dynamical systems have been long used to study various physical and natural systems occurring around us. Many of these processes have been modeled using discrete or continuous systems and their long term behavior has been investigated. The theory has found applications in a variety of fields such as complex systems, control theory, biomechanics and cognitive sciences. While [3] used dynamical systems theory to study the agent environment interaction in the cognitive setting, in [8] the authors used dynamical systems approach to lower extremity running injuries. In [11], the authors used the theory of discrete systems to model the chemical turbulence in a system. Although, the theory has been used extensively in a variety of fields, most of the cases have been approximated using an autonomous system. Consequently, the governing rule f is assumed to be constant (with time) and the dynamics of the system (X, f) is used to approximate

* Corresponding author.
E-mail addresses: puneet@iitj.ac.in (P. Sharma), manishrghv@gmail.com (M. Raghav).

https://doi.org/10.1016/j.topol.2018.07.014
0166-8641/

the dynamics of the underlying system. Although such studies have resulted in good approximations of the underlying systems, better approximations can be obtained by allowing the governing rule to be time variant. Thus there is a strong need to develop the theory of non-autonomous dynamical systems. Some investigations for such a setting in the discrete case have been made and interesting results have been obtained. While [9] study the topological entropy when the family $\mathbb{F}$ is equicontinuous or uniformly convergent, [10] discusses minimality conditions for a non-autonomous system on a compact Hausdorff space while focusing on the case when the non-autonomous system is defined on a compact interval of the real line. In [12], the authors investigate a non-autonomous system generated by a finite family of continuous self maps. In the process, they study properties such as transitivity, weak mixing, topological mixing, existence of periodic points, various forms of sensitivities and Li–Yorke chaos. In [7], the authors prove that if $f_n \to f$, in general there is no relation between chaotic behavior of the non-autonomous system generated by f_n and the chaotic behavior of f. In [2], the authors investigate properties such as weakly mixing, topological mixing, topological entropy and Li–Yorke chaos for the non-autonomous system. Before we move further, we give some of the basic concepts and definitions required.

Let (X, d) be a compact metric space and let $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ be a family of continuous self maps on X. For any initial seed x_0, any such family generates a non-autonomous dynamical system via the relation $x_n = f_n(x_{n-1})$. Throughout this paper, such a dynamical system will be denoted by $(X, \mathbb{F})$. For any $x \in X$, $\{f_n \circ f_{n-1} \circ \ldots \circ f_1(x) : n \in \mathbb{N}\}$ defines the orbit of x. For notational convenience, let $\omega_{n+k}^n = f_{n+k} \circ f_{n+k-1} \circ \ldots \circ f_{n+1}$ and $\omega_n(x) = f_n \circ f_{n-1} \circ \ldots \circ f_1(x)$ (the state of the system after n iterations).

A point x is called *periodic* for $(X, \mathbb{F})$ if there exists $n \in \mathbb{N}$ such that $\omega_{nk}(x) = x$ for all $k \in \mathbb{N}$. The least such n is known as the period of the point x. A system $(X, \mathbb{F})$ is called *feebly open* if for any non-empty open set U in X, $int(f(U)) \neq \emptyset$ for all $f \in \mathbb{F}$. The system $(X, \mathbb{F})$ is *equicontinuous* if for each $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(\omega_n(x), \omega_n(y)) < \epsilon$ for all $n \in \mathbb{N},\ x, y \in X$. The system $(X, \mathbb{F})$ is *transitive* (or $\mathbb{F}$ is transitive) if for each pair of non-empty open sets U, V in X, there exists $n \in \mathbb{N}$ such that $\omega_n(U) \bigcap V \neq \emptyset$. The system $(X, \mathbb{F})$ is said to be *minimal* if every point of X has dense orbit in X. The system $(X, \mathbb{F})$ is said to be *weakly mixing* if for any collection of non-empty open sets U_1, U_2, V_1, V_2 in X there exists a natural number n such that $\omega_n(U_i) \bigcap V_i \neq \emptyset$, $i = 1, 2$. Equivalently, we say that the system is weakly mixing if $\mathbb{F} \times \mathbb{F}$ is transitive. The system is said to be *topologically mixing* if for every pair of non-empty open sets U, V there exists a natural number K such that $\omega_n(U) \bigcap V \neq \emptyset$ for all $n \geq K$. The system is said to be *sensitive* if there exists a $\delta > 0$ such that for each $x \in X$ and each neighborhood U of x, there exists $n \in \mathbb{N}$ such that $diam(\omega_n(U)) > \delta$. If there exists $K > 0$ such that $diam(\omega_n(U)) > \delta\ \forall n \geq K$, then the system is *cofinitely sensitive*. A pair (x, y) is proximal for $(X, \mathbb{F})$ if $\liminf\limits_{n\to\infty} d(\omega_n(x), \omega_n(y)) = 0$. For any $x \in X$, the set $Prox_{\mathbb{F}}(x) = \{y : (x, y)$ is proximal for $(X, \mathbb{F})\}$ is called the proximal cell of x in $(X, \mathbb{F})$. A system $(X, \mathbb{F})$ is said to exhibit dense set of proximal pairs if the set of pairs proximal for $(X, \mathbb{F})$ is dense in $X \times X$. A pair of points x, y in X is called a δ-Li–Yorke pair if $\limsup\limits_{n\to\infty} d(\omega_n(x), \omega_n(y)) > \delta$ but $\liminf\limits_{n\to\infty} d(\omega_n(x), \omega_n(y)) = 0$. A system $(X, \mathbb{F})$ is Li–Yorke sensitive if there exists $\delta > 0$ such that for each $x \in X$ and each neighborhood U of x there exists $y \in U$ such that (x, y) is a δ-Li–Yorke pair. For any $x \in X$, let $LY_{\mathbb{F}}(x) = \{y \in X : (x, y)$ is a Li–Yorke pair for $(X, \mathbb{F})\}$ is called the Li–Yorke cell of x. It may be noted that in case the f_n's coincide, the above definitions coincide with the known notions of an autonomous dynamical system [4–6]. Some basic concepts and recent works in this area can be found in literature [1,2,7,9,10,12].

Let X be a compact space and let $\mathcal{K}(X)$ denote the collection of all non-empty compact subsets of X. For any $A, B \in \mathcal{K}(X)$ define $D_H(A, B) = \inf\{\epsilon > 0 : A \subset S(B, \epsilon)$ and $B \subset S(A, \epsilon)\}$ where $S(A, \epsilon) = \bigcup\limits_{x \in A} S(x, \epsilon)$ is the ϵ-ball around A. Then, D_H defines a metric on $\mathcal{K}(X)$ (known as the Hausdorff metric). It is known that a system (X, f) is a weakly mixing (topologically mixing) if and only if for any compact set K with non-empty interior $\limsup\limits_{n\to\infty} f^n(K) = X$ ($\lim\limits_{n\to\infty} f^n(K) = X$) with respect to the metric D_H.