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Semicontinuity of betweenness functions

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ABSTRACT

A ternary relational structure $\langle X, [\cdot,\cdot,\cdot]\rangle$, interpreting a notion of betweenness, gives rise to the family of intervals, with interval $[a,b]$ being defined as the set of elements of X between a and b. Under very reasonable circumstances, X is also equipped with some topological structure, in such a way that each interval is a closed nonempty subset of X. The question then arises as to the continuity behavior—within the hyperspace context—of the betweenness function $\{x,y\} \mapsto [x,y]$. We investigate two broad scenarios: the first involves metric spaces and Menger's betweenness interpretation; the second deals with continua and the subcontinuum interpretation.

1. Introduction and preliminaries

Let $\langle X, [\cdot,\cdot,\cdot]\rangle$ be a **ternary structure**; i.e., X is a set and $[\cdot,\cdot,\cdot] \subseteq X^3$ is a ternary relation on X. The relation is intended to convey a notion of inclusive betweenness, so we assume it to be **basic**; i.e., it satisfies

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the conditions that $[a, a, b]$ and $[a, b, b]$ always hold (*inclusivity*), that $[a, c, b]$ implies $[b, c, a]$ (*symmetry*), and that $[a, c, a]$ implies $a = c$ (*uniqueness*).

For each $a, b \in X$, we define the **interval** $[a, b]$ to be the set $\{x \in X : [a, x, b]\}$. Then, in interval terms, the three basic criteria above become $[a, b] \supseteq \{a, b\}$, $[a, b] = [b, a]$, and $[a, a] = \{a\}$, respectively. There is a unique smallest basic relation, namely the one where $[a, b] = \{a, b\}$ identically. This we refer to here as the **minimal** ternary relation on X.

The points a and b are **bracket points** (and $\{a, b\}$ a **bracket pair**) for the interval $[a, b]$. If I is an interval, its **bracket set** is defined to be $\{\{a, b\} : [a, b] = I\}$.

The assignment $\{x, y\} \mapsto [x, y]$ is the **betweenness function** associated with $[\cdot, \cdot, \cdot]$, and is denoted throughout the text by $[\cdot, \cdot]$. Hence the bracket set for interval I is just the fiber over I with respect to this function.

The present paper is a continuation of the project initiated in [2] (see also [3,4]); here we are interested in the issue of when nearby bracket pairs give rise to nearby intervals. The best way to make sense of this is to give X some topological structure, and inquire into whether the betweenness function is continuous in the context of hyperspaces [15].

We consider two broad case studies: the first is where X is a metric space, and $[a, c, b]$ means that c lies between a and b in the sense of Menger [14]; the second is where X is a continuum, and $[a, c, b]$ means that c lies in every subcontinuum of X that contains $\{a, b\}$. In the first study it is both the topology and the geometry of metric spaces that dictate the continuity of the betweenness function; in the second it is the topology alone of (not necessarily metrizable) continua.

For a topological space X, we denote by 2^X (resp., $\mathcal{K}(X)$) its *hyperspace* of all nonempty closed (resp., nonempty closed connected) subsets. If U is an open set in X, U^+ (resp., U^-) denotes the set $\{C \in 2^X : C \subseteq U\}$ (resp., $\{C \in 2^X : C \cap U \neq \emptyset\}$). The **upper** (resp., **lower**) **Vietoris topology** on 2^X is subbasically generated by sets of the form U^+ (resp., U^-), as U ranges over the open subsets of X. The join of these two topologies is the **Vietoris topology** on 2^X, and we view $\mathcal{K}(X)$ as inheriting this topology.

We let $\omega := \{0, 1, 2, \dots\}$ denote the set of finite ordinals. It will be convenient to eliminate zero at times, so we use the symbol $\mathbb{N}$ to denote $\omega \setminus \{0\}$.

For each $n \in \mathbb{N}$, let $\mathcal{F}_n(X)$ denote the n**-fold symmetric power** of X, the hyperspace consisting of those $C \in 2^X$ with at most n elements (also equipped with the inherited Vietoris topology). When X is a T_1 space, the function $x \mapsto \{x\}$ defines a homeomorphism from X onto $\mathcal{F}_1(X)$ (where the inherited upper and lower Vietoris topologies coincide); when X is Hausdorff, each $\mathcal{F}_n(X)$ is a closed subspace of 2^X. If X is also normal, then $\mathcal{K}(X)$ is closed in 2^X as well. Of the hyperspaces $\mathcal{F}_n(X)$, we will be interested only in the case $n = 2$ from here on.

The following is a simple, but useful, result (see, e.g., [15]).

Lemma 1.1. *The Vietoris topology on 2^X is basically generated by sets of the form $[\![U_1, \dots, U_n]\!] := \{C \in 2^X : C \subseteq U_1 \cup \dots \cup U_n$ and $C \cap U_i \neq \emptyset$ for $1 \leq i \leq n\}$, where $n \in \mathbb{N}$ and $\langle U_1, \dots, U_n \rangle$ ranges over all n-tuples of open subsets of X.*

Proof. This is a direct consequence of the following identities: $U^+ \cap V^+ = [\![U \cap V]\!]$, $U^- \cap V^- = [\![X, U, V]\!]$, $U^+ \cap V^- = [\![U, U \cap V]\!]$, and $[\![U_1, \dots, U_n]\!] = (\bigcup_{i=1}^n U_i)^+ \cap (\bigcap_{i=1}^n U_i^-)$. $\square$

Unless specified otherwise, the default topology on the hyperspaces defined above is the Vietoris topology. It is a basic fact about this topology (see [15, §4]) that X is compact Hausdorff (resp., compact metrizable) if and only if the same is true for any of these hyperspaces.

If X and Y are two topological spaces, a function $\varphi : Y \to 2^X$ is **upper** (resp., **lower**) **semicontinuous** (**usc** and **lsc**, respectively) at $a \in Y$ if it is continuous at a in the usual sense for the upper (resp., lower) Vietoris topology on 2^X. So φ is continuous at a if and only if it is both usc and lsc at a. And when we unpack the definitions, we see that φ is usc (resp., lsc) at a just in case for any open $U \subseteq X$ such that $\varphi(a) \subseteq U$ (resp.,