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Metrizability of the space of quasicontinuous functions

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ABSTRACT

Let X be a topological space, (Y, d) be a metric space, $Q(X, Y)$ be the space of quasicontinuous functions from X to Y and τ_{UC} be the topology of uniform convergence on compacta. We study first countability, metrizability and complete metrizability of $(Q(X, Y), \tau_{UC})$. We will apply our results to characterize sequentially compact subsets of $(Q(X, Y), \tau_{UC})$.

1. Introduction

Quasicontinuous functions were introduced by Kempisty in 1932 in [15]. They were studied in many papers, see for example [2], [16], [21] and others. Quasicontinuous functions are important in many areas of mathematics. They found applications in the study of minimal usco and minimal cusco maps [8,10], in the study of topological groups [3,18,20], in proofs of some generalizations of Michael's selection theorem [6], in the study of extensions of densely defined continuous functions [7], in the study of dynamical systems [4]. The quasicontinuity is also used in the study of CHART groups [19].

2. Preliminaries

In what follows let X, Y be Hausdorff topological spaces, $\mathbb{N}$ be the set of positive natural numbers, $\mathbb{R}$ be the space of real numbers with the usual metric. The symbol $\overline{A}$ will stand for the closure of the set A in a topological space.

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A function $f : X \to Y$ is quasicontinuous [21] at $x \in X$ if for every open set $V \subset Y$, $f(x) \in V$ and every open set $U \subset X$, $x \in U$ there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

We say that a subset of X is quasi-open (or semi-open) [21] if it is contained in the closure of its interior. Then a function $f : X \to Y$ is quasicontinuous if and only if $f^{-1}(V)$ is quasi-open for every open set $V \subset Y$.

Denote by $F(X, Y)$ the set of all functions from X to Y and by $Q(X, Y)$ the set of all quasicontinuous functions in $F(X, Y)$.

By $\mathfrak{K}(X)$ we mean the family of all nonempty compact subsets of X.

Let (Y, d) be a nontrivial metric space.

The topology of uniform convergence on compact sets on $F(X, Y)$ we denote by τ_{UC}. This topology is induced by the uniformity $\mathfrak{U}_{UC}$ which has a base consisting of sets of the form

$$W(K, \varepsilon) = \{(f, g) : \ \forall\, x \in K \ \ d(f(x), g(x)) < \varepsilon\},$$

where $K \in \mathfrak{K}(X)$ and $\varepsilon > 0$. The general τ_{UC}-basic neighbourhood of $f \in F(X, Y)$ will be denoted by $W(f, K, \varepsilon)$, i.e. $W(f, K, \varepsilon) = W(K, \varepsilon)[f]$.

3. Metrizability

A topological space X is hemicompact [5] if in the family of all compact subspaces of X ordered by the inclusion there exists a countable cofinal subfamily.

Every hemicompact space is σ-compact, but not vice versa. The space of rationals with the usual topology is a σ-compact space which is not hemicompact. A locally compact σ-compact space is hemicompact.

Theorem 3.1. *Let X be a topological space and (Y, d) be a metric space. Then the following are equivalent:*

(1) *The uniformity $\mathfrak{U}_{UC}$ on $Q(X, Y)$ is induced by a metric;*
(2) *$(Q(X, Y), \tau_{UC})$ is metrizable;*
(3) *$(Q(X, Y), \tau_{UC})$ is first countable;*
(4) *X is hemicompact.*

Proof. (4) ⇒ (1) Let $\{K_n : \ n \in \mathbb{N}\}$ be a countable cofinal subfamily in $\mathfrak{K}(X)$ with respect to the inclusion. The family

$$W(K, \varepsilon) = \{(f, g) : \ \forall\, x \in K \ \ d(f(x), g(x)) < \varepsilon\}$$

where $K \in \mathfrak{K}(X)$ and $\varepsilon > 0$, is a base of $\mathfrak{U}_{UC}$ on $Q(X, Y)$. Since for every $K \in \mathfrak{K}(X)$ there is $n \in \mathbb{N}$ with $K \subset K_n$, the family

$$\{W(K_n, \frac{1}{m}) : \ n, m \in \mathbb{N}\}$$

is a countable base of $\mathfrak{U}_{UC}$. Thus by the metrization theorem in [14] $(Q(X, Y), \mathfrak{U}_{UC})$ is metrizable.

(1) ⇒ (2) and (2) ⇒ (3) are obvious.

(3) ⇒ (4) Let $y_0 \in Y$ and let f be the constant function on X mapping each point to y_0. By the assumption f has a countable base $\{W(f, K_n, \varepsilon_n) : \ n \in \mathbb{N}\}$. We claim that $\{K_n : \ n \in \mathbb{N}\}$ is a countable cofinal family in $\mathfrak{K}(X)$ with respect to the inclusion. Suppose that this is not true. Thus there is $K \in \mathfrak{K}(X)$ such that for each $n \in \mathbb{N}$ there is $k_n \in K \setminus K_n$. For every $n \in \mathbb{N}$ there is an open neighbourhood U_n of k_n