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Boolean loops with compact left inner mapping groups are profinite

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A compact, Hausdorff and totally disconnected loop is called a *boolean loop*. In this note, we prove that a boolean loop with compact left inner mapping group is profinite.

1. Introduction

A groupoid L with identity $1 \in L$ is called a *loop* if the equations $Xx = y$ and $xX = y$, where X is unknown in the equation, has the unique solution in L for all $x, y \in L$. In notation we write it as $X = y/x$ and $X = x\backslash y$ respectively. In the following, we adopt the notations from [1].

Let L be a loop. For each $a \in L$ the right translation $\rho_a : L \to L$ and the left translation $\lambda_a : L \to L$ defined by $\rho_a(x) = xa$ and $\lambda_a(x) = ax$ are bijective maps on L. Then the group $M_l(L)$ generated by λ_a is called as the *left multiplication group* of L. A mapping $\alpha \in M_l(L)$ such that $\alpha(1) = 1$ is called an *left inner mapping*. The set of all left inner mappings forms a subgroup of $M_l(L)$, which is called the *left inner mapping group* of L and is denoted by $D(L)$. One can check that $D(L) = \langle\{\delta_{a,b} \mid a, b \in L\}\rangle$, where $\delta_{a,b} = \lambda_{ab}^{-1}\lambda_a\lambda_b$. This defines a map $\delta : L \times L \to D(L)$ given by $(a, b) \mapsto \delta_{a,b}$. Also, the set $\lambda(L) = \{\lambda_a \mid a \in L\}$ is a left transversal of $D(L)$ in $M_l(L)$. If we identify L with $\lambda(L)$, then we say L is a left transversal of $D(L)$ in $M_l(L)$.

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Due to the construction of L.V. Sabinin (see [2], [3]), the group $M_l(L)$ can be identified with the group $L \times D(L)$ with respect to the binary operation is defined as $(a,\alpha)(b,\beta) = (a\alpha(b), \delta_{a,\alpha(b)}\chi(b,\alpha)\alpha\beta)$, where $\chi : L \times D(L) \to D(L)$ is given by $\chi(a,\alpha) = \lambda^{-1}_{\alpha(a)}\alpha\lambda_a\alpha^{-1}$ (see [1, 2.14, p. 40]).

Conversely, if L is a left transversal of a subgroup Ω in a group G. Then L is a left loop with respect to the binary operation $\circ$ defined as follows. Let $a, b \in L$, then there exist unique $a \circ b \in L$ and $d_{a,b} \in \Omega$ such that $ab = (a \circ b)d_{a,b}$. If a left transversal L of a subgroup Ω in a group G is loop with respect to the above operation, then L is a called *loop transversal*. If θ is the permutation representation of the group G on L and $\langle L\rangle = G$, then $\theta(G) = M_l(L)$ and $\theta(\Omega) = D(L)$ (see [1, Theorem 2.8, p. 32]).

A non-empty subset T of a loop L is called a *subloop* of L, if it is loop with respect to induced binary operation on T. An equivalence relation R on a loop L is called a *congruence* in L, if it is a subloop of $L \times L$. Also an *invariant subloop* of a loop L is precisely the equivalence class of the identity of a congruence in L. If T is an invariant subloop of L, then the set $L/T = \{T \circ x | x \in L\}$ becomes a loop called as *quotient of* L *mod* T. Let R be the congruence associated to an invariant subloop T of L. Then we also denote L/T by L/R.

A loop L is called *topological* if L is a topological space and the binary operations $\circ$, $/$ and $\backslash$ are continuous. An inverse limit of an inverse system of finite loops with discrete topology is called a *profinite loop*. If G is compact, Hausdorff and totally disconnected group, then G is a profinite group. A compact, Hausdorff and totally disconnected loop is called a *boolean loop*. The profinite loops are boolean loops. It is not known whether a boolean loop is profinite or not. However, in [4] it is shown that a boolean loop with finite left inner mapping group is profinite. In this note, we prove the following.

Theorem 1.1. *(Main Theorem) A boolean loop with the compact left inner mapping group is profinite.*

2. Proof of the Main Theorem

Let us first mention few results which will be used to prove the main theorem.

Proposition 2.1. *([5, Proposition 2.1, p. 39]) An equivalence relation V on the loop L is a congruence on L if and only if the subset V of $L \times L$ is invariant under the diagonal action of the multiplication group G on $L \times L$.*

Proposition 2.2. *([6, Theorem 79, p. 40]) Let L be a topological loop. Let H be a subloop of L. If H is open, then H is also closed. If H is closed and has finite index, then H is open. If L is compact and H is open, then H has finite index.*

It is observed in the previous section that the left transversal and left loop are same in the sense discussed above. Let G be a Hausdorff topological group and L a loop transversal of a closed subgroup Ω in G. Then, two topologies are induced on L. One is the subspace topology τ_s induced on subset $L \subseteq G$ and other is the quotient topology τ_q obtained by the quotient space G/Ω. One can clearly observe that $\tau_q \subseteq \tau_s$. If in addition L is compact and Hausdorff, then $\tau_q = \tau_s$, for the compact Hausdorff topology is minimal Hausdorff topology. We now prove the following.

Proposition 2.3. *Let L be a compact Hausdorff topological loop. Then, the left multiplication group is a topological group with respect to the product topology on $L \times D(L)$.*

Proof. Let L be a compact Hausdorff topological loop. Then, the group $Hom(L)$ of homeomorphisms of L is a topological loop with compact open topology. This implies that $D(L) \subseteq Hom(L)$ is a topological group. Since λ_a and λ_a^{-1} are continuous, the maps δ and χ are continuous. This shows that the binary operation on $M_l(L)$ is continuous. Also, $(a, \alpha^{-1}) = (\alpha^{-1}(a'), \alpha^{-1}\chi(\alpha^{-1}(a'), \alpha)^{-1}\delta_{a,a'})$, where $a' = \lambda_a^{-1}(1)$. Therefore,