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A generalization of the rectangle condition

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ABSTRACT

In this paper, we define the *connecting rectangle condition* to check whether or not a Heegaard splitting is strongly irreducible which is a variation of the rectangle condition by Casson and Gordon. We define a *general version* of the rectangle condition. Moreover, with a similar condition defined on an n-bridge decomposition, we can check whether or not the Hempel distance of an n-bridge decomposition is greater than or equal to two.

1. Introduction

For a given closed oriented 3-manifold M, a Heegaard splitting of M is a decomposition of M into two handlebodies V and W. That is, we have the 3-manifold M by gluing V and W along an orientation reversing self-homeomorphism of $S(= V \cap W = \partial V = \partial W)$. Then M is denoted by $V \cup_S W$. A Heegaard splitting $V \cup_S W$ is *strongly irreducible* if ∂D_1 meets ∂D_2 in S for any pair of essential disks $D_1(\subset V)$ and $D_2(\subset W)$. Casson and Gordon [2] introduced the *rectangle condition* on Heegaard surfaces to show strong irreducibility of Heegaard splittings of 3-manifolds. By referring to Takao's work [7], the author [5] defined the *rectangle condition* on bridge spheres for n-bridge decompositions of knots or links. Moreover, the satisfaction of the rectangle condition can guarantee that the Hempel distance is greater than or equal to two. However, when we apply the rectangle condition to check whether or not the Heegaard splitting of a 3-manifold is strongly irreducible, or there exists a perturbation with respect to the bridge sphere for an n-bridge decomposition of a link, there is a checklist which has many elements depending on n. So, the purpose of this paper is to find a better criterion which keeps the same property but less elements to be checked. In this paper, we introduce a new condition, named the *connecting rectangle condition*, which has a checklist having less elements but the satisfaction of the new condition still can guarantee that the Hempel distance is greater than or equal to two. For instance, we need 32 elements in the checklist for the satisfaction of the new condition in case of a genus three Heegaard, while the classic condition needs 81 elements to be checked for the same Heegaard splitting. In spite of this advantage of the connecting rectangle condition, we are not able to say that the connecting rectangle condition is, in general, better than the rectangle condition on the

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Heegaard splitting of 3-manifolds since the basic settlement of the two criteria are different. The rectangle condition is a criterion based on pants decompositions of a Heegaard surface, while the connecting condition is a criterion based on *systems* of separating disks which separate the given handlebodies of the Heegaard splitting into two 3-dimensional balls. Whereas, two criteria on n-bridge decomposition are more closely related. The rectangle condition on n-bridge decompositions is a criterion based on a pants decomposition of a sphere with n boundary components which is obtained from the bridge sphere by removing n "*special*" two punctured disks. However, we note that the pants decomposition of the sphere with n boundary components is not necessary for the main property of the rectangle or connecting rectangle condition because the system of special two punctured disks already have a meaningful information. So, it is enough to investigate the n *special* simple closed curves in the $2n$-punctured sphere which are bounded by the n special two punctured disks.

Hempel distance is a measurement of complexity for Heegaard splittings of 3-manifolds [3]. In particular, if the Hempel distance of a Heegaard splitting is greater than or equal to two then the Heegaard splitting is strongly irreducible by the definition. Bachman and Schleimer [1] transferred the definition from Heegaard surfaces to n-bridge spheres to compute the complexity of n-bridge decompositions. We show that if a knot K satisfies the connecting rectangle condition for an n-bridge decomposition $(T_1, T_2; S^2)$ of K then the Hempel distance for the n-bridge decomposition is greater than or equal to 2. Moreover, this can guarantee that K is not perturbed. (Refer to [7].) Otal [6] pointed out that any n-bridge presentation of a 2-bridge knot is perturbed for $n > 2$. Therefore, if a knot K has a 3-bridge decomposition $(T_1, T_2; S^2)$ which satisfies the connecting rectangle condition then K is a 3-bridge knot. To demonstrate the effectiveness of the connecting rectangle condition we would give a specific example on n-bridge decomposition of links which satisfy the connecting rectangle condition. We note that we also can construct a strongly irreducible Heegaard splitting which satisfies the connecting rectangle condition by using the branched double covering of the example on n-bridge decomposition of links.

2. Rectangle condition and connecting rectangle condition on Heegaard splittings

We first introduce the *rectangle condition* by Casson and Gordon. Let V be a genus $g(> 1)$ handlebody and let $S = \partial V$. We can choose a collection of $3g-3$ essential disks $\{E_1, ..., E_{3g-3}\}$, called an *extended system of disks*, so that they are pairwise non-isotopic and disjoint, and they cut V into $2g-2$ balls $V_1, V_2, ..., V_{2g-2}$. That is, $V_1, ..., V_{2g-2}$ are the closures of the $2g-2$ components of $V - \cup_{i=1}^{3g-3} E_i$. Let $P_i = V_i \cap \partial V$. We note P_i is a pair of pants and $P_1 \cup \cdots \cup P_{2g-2}$ is a pants decomposition of S. Let P and Q be pairs of pants. We say the P and Q are *tight* if (1) there is no bigon Δ in P and Q with $\partial\Delta = \alpha \cup \beta$, where α is a subarc of ∂P and β is a subarc of ∂Q, and (2) for any two components of ∂P and any two components of ∂Q, there exists a rectangle R embedded in P and Q such that the interior of R is disjoint from $\partial P \cup \partial Q$ and the four edges of ∂R are subarcs of the selected four boundaries respectively. Now, let $\mathcal{P}$ and $\mathcal{Q}$ be two pants decompositions of S. We say that $\mathcal{P}$ and $\mathcal{Q}$ satisfy the *rectangle condition* if P_i and Q_j are tight for any $i, j \in \{1, 2, ..., 2g-2\}$. Proposition 2.1 explains the property we focus on.

Proposition 2.1. *(Casson, Gordon [2]) Suppose $\mathcal{P}$ and $\mathcal{Q}$ are pants decompositions of S which are obtained by removing extended systems of disks in V and W respectively. If $\mathcal{P}$ and $\mathcal{Q}$ satisfy the rectangle condition, then the Heegaard splitting $V \cup_S W$ is strongly irreducible.*

Now, we consider a collection of $g+1$ essential disks $\{D_1, ..., D_{g+1}\}$ in V so that they are pairwise non-isotopic and disjoint, and they cut V into only two balls V_1 and V_2. That is, V_1 and V_2 are two closures of the components of $V - \cup_{i=1}^{g+1} D_i$ in V. We note that the collection contains at most one separating essential disk. Let $X_i = V_i \cap \partial V$. The collection of essential disk $\{D_1, ..., D_{g+1}\}$ is called a *system of evenly separable disks* if X_i is a planar surface with $g+1$ boundary components. We say that $\{X_1, X_2\}$ is a *double(g-legged)*