Topology and its Applications 241 (2018) 162–171

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Embedding products into symmetric products of finite graphs

Florencio Corona-Vázquez, Russell Aarón Quiñones-Estrella *,
Javier Sánchez-Martínez, Hugo Villanueva

Universidad Autónoma de Chiapas, Facultad de Ciencias en Física y Matemáticas, Carretera Emiliano Zapata Km. 8, Rancho San Francisco, Terán, C.P. 29050, Tuxtla Gutiérrez, Chiapas, Mexico

A R T I C L E I N F O

Article history:
Received 3 April 2018
Accepted 4 April 2018
Available online 5 April 2018

MSC:
54B10
54B20
54C25
54F15

Keywords:
Continuum
Embedding
Finite graph
Product
Symmetric products

A B S T R A C T

For each positive integer n and a continuum X, we will denote by $F_n(X)$ the nth-symmetric product of X and by X^n the product of X with itself n times. In this paper we study finite graphs X such that X^n can be embedded in $F_n(X)$. We also present a geometric model of the third symmetric product of a simple triod.

1. Introduction

A *continuum* means a nonempty, compact, connected metric space. Given a continuum X and a positive integer n, we denote by X^n the product of X with itself n times with the product topology and by $F_n(X)$ the hyperspace of all nonempty subsets of X with at most n points, endowed with the Hausdorff metric (see [8, Definition 0.1, p. 1]), this is the so called *nth-symmetric product of* X. It is known that $F_n(X)$ is a continuous image of X^n (see [1, p. 877]). In [4] the authors studied the problem of determining continua X such that X^n can be embedded in $F_n(X)$, they proved that if X is a finite graph then X^2 can be embedded into $F_2(X)$ if and only if X is an arc. In Section 3, we show some results for $n \geq 3$, in this direction, Theorem 3.6 is the main result of the section:

If X is a finite graph then X^3 can be embedded into $F_3(X)$ if and only if X is an arc.

* Corresponding author.
E-mail addresses: florencio.corona@unach.mx (F. Corona-Vázquez), rusell.quinones@unach.mx (R.A. Quiñones-Estrella), jsanchezm@unach.mx (J. Sánchez-Martínez), hugo.villanueva@unach.mx (H. Villanueva).

https://doi.org/10.1016/j.topol.2018.04.003
0166-8641/

In Section 4, we give a positive answer to a question asked by E. Castañeda and J. Sánchez in [4, Question 4.14, p. 205]. In [6, pages 55 and 56] it is commented that E. Castañeda found a model for $F_3(Y)$, where Y is a simple triod. It says that Castañeda showed that $F_3(Y)$ is the cone over a torus with four disks attached to it, one as an "equator" and the three other ones as "meridians" (see Figure 24 in [6]). In Section 5 of this paper we show that the model proposed by Castañeda is wrong. We construct a correct model, the difference with Castañeda proposal is that one must change the torus by a Klein bottle, with the four disks attached in a similar way.

2. Preliminaries

By a *finite graph* we mean a continuum X which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both of their end points. Given a positive integer n, a *simple n-od* is a finite graph, denoted by T_n, which is the union of n arcs emanating from a single point, v, and otherwise disjoint from one another. The point v is called the *vertex* of the simple n-od. A simple 3-od, T_3, will be called a *simple triod*. An n-cell is a space homeomorphic to $[0,1]^n$.

Given a finite graph X, $p \in X$ and a positive integer n, we say that *p is of order n in X*, denoted by $ord(p, X) = n$, if p has a closed neighborhood which is homeomorphic to a simple n-od having p as the vertex. If $ord(p, X) = 1$, then p has a neighborhood which is an arc having p as one of its end points and we will call it an *end point of X*. If $ord(p, X) = 2$, then p has a neighborhood which is an arc, p is not an end point of it, and we will call it an *ordinary point of X*. A point $p \in X$ is a *ramification point of X* if $ord(p, X) \geq 3$. The *vertices* of a finite graph X will be the end points and the ramification points of X. An *edge* will be an arc joining two vertices of X and having exactly two vertices of X. The set of all ramification points of X will be denoted by $R(X)$.

For a positive integer m, a *complete graph*, denoted by K_m, is a finite graph with exactly m vertices such that any two vertices are joined by exactly one of its edges. Let V be the set of vertices of a finite graph X, we say that X is *bipartite* if there exist two nonempty subsets V_1 and V_2 of V such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and each edge of X joins a vertex of V_1 with a vertex of V_2. A bipartite graph X is said to be *complete bipartite* if each vertex of V_1 is joined to every vertex of V_2 by edges of X. Given two positive integers m and n, $K_{m,n}$ will denote a complete bipartite graph such that $|V_1| = m$, $|V_2| = n$. A *simple closed curve* is any space homeomorphic to $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. In general, for a positive integer n, S^n will denote the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.

Given a topological space X, the *topological cone of X*, denoted by $\mathrm{cone}(X)$, is the quotient space obtained from $X \times [0,1]$ by shrinking $X \times \{1\}$ to a point.

Given positive integers m and n, and nonempty subsets $K_1, \ldots, K_m$ of a continuum X, we will denote by $\langle K_1, \ldots, K_m \rangle_n$ the set

$$\left\{ A \in F_n(X) : A \subset \bigcup_{i=1}^{m} K_i \text{ and for each } i \in \{1, \ldots, m\},\ A \cap K_i \neq \emptyset \right\}.$$

For a positive integer n, it is known that the sets of the form $\langle U_1, \ldots, U_m \rangle_n$, where m is a positive integer and each set U_i is open in X, form a basis for the topology of $F_n(X)$ called the *Vietoris topology* (see [8, Theorem 0.11, p. 9]), and that the Vietoris topology and the topology induced by the Hausdorff metric are the same (see [8, Theorem 0.13, p. 9]).

3. Embedding X^n into $F_n(X)$

First, we consider the arc, the simple closed curve and simple n-ods. It is known that $[0,1]^n$ is embedded into $F_n([0,1])$ for each positive integer n. For the simple closed curve we have the following.