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The strength of prime separation, sobriety, and compactness theorems

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A R T I C L E   I N F O

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A B S T R A C T

We investigate in **ZF** set theory without choice principles a general lattice-theoretical prime separation lemma and compare it with diverse statements about variants of the sobriety concept for topological spaces. Some of these properties coincide in the presence of choice principles but differ in their absence. **UP**, the Ultrafilter Principle (or, equivalently, the Prime Ideal Theorem) is not only equivalent to the Separation Lemma, but also necessary and sufficient for the desired coincidences. Furthermore, we prove the equivalence of **UP** to several statements about filtered systems of compact sets, among them the Hofmann–Mislove Theorems, several compact intersection theorems, and an irreducible transversal theorem. Moreover, many fundamental dualities between certain categories of topological spaces and categories of ordered structures turn out to be equivalent to **UP**. But we also give choice-free proofs for such dualities, amending slightly the involved definitions.

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0. Introduction

In the present paper our strict demand is to work in Zermelo–Fraenkel set theory (**ZF**) without any unprovable choice principles, if not otherwise stated. Our approach to diverse topological, order-theoretical and algebraic problems and their solutions will be the following: given a certain mathematical property P that plays a role in theorems whose proofs require choice principles, we pass to a property P′ that makes these theorems provable in **ZF** without choice and is equivalent to the original property P if certain choice principles are assumed. We say then an object has property P *strictly* if it has property P′. In that spirit, we introduce *strictly spatial lattices*, *strictly sober spaces*, and *strictly continuous posets*.

Many mathematical statements, classically derived from the Axiom of Choice (**AC**), are already consequences of – or even equivalent to – the weaker Ultrafilter Theorem or Ultrafilter Principle (**UP**), asserting that any proper set-theoretical filter is contained in an ultrafilter. Famous examples are compactness theorems of mathematical logic (see [41]). In the present paper we demonstrate that diverse purely topological compactness theorems, many order-topological dualities, and the coincidence of various topological sobriety concepts, are all equivalent to **UP**.

(Scott) open filters in posets or lattices are down-directed upper sets having a complement that is closed under up-directed joins; they will play a central role in our study. Referring to the poset δP of all open filters in a poset P, we say P is *δ-separated* if any principal ideal disjoint from an open filter is contained in a prime principal ideal still disjoint from that filter. A complete lattice is called *strictly spatial* if it is not only δ-separated but also spatial, i.e. isomorphic to the lattice of open sets of a topological space. In Section 1, we start with an investigation of the *Separation Lemma*, which is equivalent to **UP** and says that every distributive complete lattice is δ-separated. Moreover, **UP** will turn out to be equivalent to the strict spatiality of various kinds of locales (frames).

In Section 2, we compare the usual concept of sobriety for topological spaces with another one that is equivalent to the former in **ZF** + **UP** but more effective in practice; we call it *strict sobriety* or *δ-sobriety*, referring to the Lawson dual $\delta\mathcal{T}$ of the open set lattice $\mathcal{T}$ of the given space: the defining condition for strict sobriety is that each $\mathcal{V} \in \delta\mathcal{T}$ contains all open neighborhoods of its intersection. Hofmann and Mislove [40] have shown that in **ZFC** = **ZF** + **AC** every sober space is strictly sober and well-filtered (that is, each neighborhood of a filtered intersection of compact saturated sets contains one of them). We shall prove in **ZF** the equivalence of these two conclusions to **UP**, while sobriety and well-filteredness follow from strict sobriety without invoking any choice principles. Fundamental is the observation that strict sobriety of a space is equivalent to strict spatiality of its topology, regarded as a complete lattice.

In Section 3, we shall establish the equivalence of **UP** to several basic topological principles concerning certain filter bases of compact sets, among them:

- **CIT** *the Compact Intersection Theorem: Any filter base of compact saturated sets in a sober space has a compact intersection.*
- **NIT** *the Non-void Intersection Theorem: Any filter base of compact saturated sets in a sober space has a non-void intersection.*
- **ITT** *the Irreducible Transversal Theorem: Any collection $\mathcal{K}$ of compact sets whose saturations form a filter base has an irreducible transversal.*

Here, a *transversal* of $\mathcal{K}$ is a subset of $\bigcup\mathcal{K}$ that meets all members of $\mathcal{K}$, and the *saturation* of a set is the intersection of all its neighborhoods. Furthermore, our study will not only reveal the equivalence of **UP** to the Hofmann–Mislove Theorems [32,40] but also provide entirely choice-free variants of these theorems. On the way through our implication circuit we use the fact that **UP** is equivalent to Tychonoff's Theorem for any class of sober spaces that contains a two-element discrete space. As a consequence of our approach via