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Minimal charts

Teruo Nagase [a], Akiko Shima [b,*,1]

[a] Tokai University, 4-1-1 Kitakaname, Hiratuka, Kanagawa, 259-1292 Japan
[b] Department of Mathematics, Tokai University, 4-1-1 Kitakaname, Hiratuka, Kanagawa, 259-1292 Japan

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ABSTRACT

In this paper, we give definitions of three kinds of minimal charts, and we investigate properties of minimal charts and establish fundamental theorems characterizing minimal charts. To classify charts with two or three crossings we use the fundamental theorems. In the future paper, we give an enumeration of the charts with two crossings.

1. Introduction

Charts are oriented labeled graphs in a disk with three kinds of vertices called black vertices, crossings, and white vertices (see page 3 for the precise definition of charts). From a chart, we can construct an oriented closed surface embedded in 4-space $\mathbb{R}^4$ (see [6, Chapter 14, Chapter 18 and Chapter 23]). A C-move is a local modification between two charts in a disk (see Section 2 for C-moves). A C-move between two charts induces an ambient isotopy between oriented closed surfaces corresponding to the two charts. Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

We will work in the PL or smooth category. All submanifolds are assumed to be locally flat. A *surface link* is a closed surface embedded in 4-space $\mathbb{R}^4$. A *2-link* is a surface link each of whose connected component is a 2-sphere. A *2-knot* is a surface link which is a 2-sphere. An orientable surface link is called a *ribbon surface link* if there exists an immersion of a 3-manifold M into $\mathbb{R}^4$ sending the boundary of M onto the surface link such that each connected component of M is a handlebody and its singularity consists of ribbon

* Corresponding author.
E-mail addresses: nagase@keyaki.cc.u-tokai.ac.jp (T. Nagase), shima@keyaki.cc.u-tokai.ac.jp (A. Shima).
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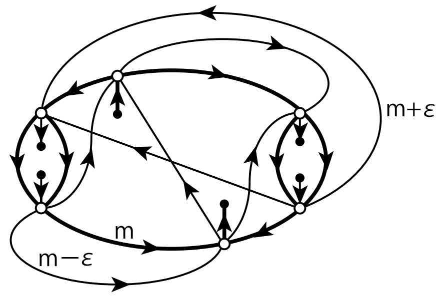

Fig. 1. The letter m is a label and $\varepsilon = \pm 1$.

singularities, here a ribbon singularity is a disk in the image of M whose pre-image consists of two disks; one of the two disks is a proper disk of M and the other is a disk in the interior of M. In the words of charts, a ribbon surface link is a surface link corresponding to a *ribbon chart*, a chart C-move equivalent to a chart without white vertices [4]. A chart is called a 2-*link chart* if a surface link corresponding to the chart is a 2-link.

In this paper, we denote the closure, the interior, the boundary, and the complement of (...) by $Cl(...)$, Int(...), $\partial(...)$, $(...)^c$ respectively. Also for a finite set X, the notation $|X|$ denotes the number of elements in X.

Kamada showed that any 3-chart is a ribbon chart [4]. Kamada's result was extended by Nagase and Hirota: Any 4-chart with at most one crossing is a ribbon chart [7]. We showed that any n-chart with at most one crossing is a ribbon chart [13]. We also showed that any 2-link chart with at most two crossings is a ribbon chart [14], [15].

Charts have strong conditions on orientations of arcs around vertices. In a small neighborhood of each white vertex, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward (see Fig. 2(c)). Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward (resp. outward) is called a *middle arc* at the white vertex. Observing precisely middle arcs, orientations of edges, and a part of a chart cutting by a disk called a *tangle*, we shall prove the following theorem [18]:

> Any 2-link chart with at most three crossings is C-move equivalent to either a ribbon chart, or the disjoint union of a ribbon chart and a chart as shown in Fig. 1 or its "reflection".

In this paper we establish fundamental theorems characterizing c-minimal charts, w-minimal charts and cw-minimal charts. For the classification theorem above, we use the fundamental theorems obtained in this paper.

For a 4-chart as shown in Fig. 1, we obtain a 2-twist spun trefoil by setting $m = 2$ (see [4, p. 144], [6, p. 170]). It is well known that the 2-knot is not a ribbon 2-knot. On the other hand, Hasegawa showed that if a non-ribbon chart representing a 2-knot is minimal, then the chart must possess at least six white vertices [2] where a minimal chart Γ means its complexity $(w(\Gamma), -f(\Gamma))$ is minimal among the charts C-move equivalent to the chart Γ with respect to the lexicographic order of pairs of integers, here $w(\Gamma)$ is the number of white vertices in Γ, $f(\Gamma)$ is the number of free edges in Γ. Here a *free edge* is an edge of Γ containing two black vertices. Nagase, Ochiai, and Shima showed that there does not exist a minimal chart with exactly five white vertices [19]. Nagase and Shima show that there does not exist a minimal chart with exactly seven white vertices [8],[9],[10],[11],[12]. Ishida, Nagase, and Shima showed that any minimal chart with exactly four white vertices is C-move equivalent to a chart in two kinds of classes [3].