Topology and its Applications 241 (2018) 197–202

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – TOPOSYM 2016

Countable successor ordinals as generalized ordered topological spaces

Robert Bonnet [a,1], Arkady Leiderman [b,*]

[a] *Laboratoire de Mathématiques, Université de Savoie, Le Bourget-du-Lac, France*
[b] *Department of Mathematics, Ben-Gurion University of the Negev, Beer Sheva, Israel*

ARTICLE INFO

Article history:
Received 26 December 2016
Received in revised form 20 August 2017
Accepted 4 December 2017
Available online 27 March 2018

MSC:
03E10
06A05
54F05

Keywords:
Linearly ordered topological spaces
Generalized ordered topological spaces
Compact spaces
Continuous images

ABSTRACT

We prove the following Main Theorem: Every continuous image of a Hausdorff topological space X is a generalized ordered space if and only if X is homeomorphic to a countable successor ordinal (with the order topology). This is a generalization of E. van Douwen's result about orderable spaces.

1. Introduction and Main Theorem

Throughout the paper all topological spaces are assumed to be Hausdorff. Recall that L is a *Linearly Ordered Topological Space* (*LOTS*) if there is a linear ordering $\leq^L$ on the set L such that a basis of the topology λ^L on L consists of all open convex subsets. The above topology λ^L is called an *order topology*.

A topological space $\langle X, \tau^X \rangle$ is called a *Generalized Ordered Space* (*GO-space*) if $\langle X, \tau^X \rangle$ is homeomorphic to a subspace of a LOTS $\langle L, \lambda^L \rangle$, that is $\tau^X = \lambda^L \restriction X := \{U \cap X \ : \ U \in \lambda^L\}$ (see [7]).

Evidently, every LOTS, and thus any GO-space, is a Hausdorff topological space, but not necessarily separable or Lindelöf. The Sorgenfrey line Z is an example of a GO-space, which is not a LOTS, and such that every subspace of Z is separable and Lindelöf (see [3]).

* Corresponding author.
E-mail addresses: bonnet@univ-savoie.fr, Robert.Bonnet@math.cnrs.fr (R. Bonnet), arkady@math.bgu.ac.il (A. Leiderman).

[1] The first listed author gratefully acknowledges the financial support he received from the Center for Advanced Studies in Mathematics of the Ben-Gurion University of the Negev.

https://doi.org/10.1016/j.topol.2018.03.032

E. van Douwen proved the following statement [2, Theorem 1.2]: Every continuous image of X is orderable iff X is compact and countable, or, in other words, every continuous image of a Hausdorff topological space X is a LOTS iff X is homeomorphic to a countable successor ordinal.

Our article is devoted to the proof of the following Main Theorem which generalizes E. van Douwen's result. Let us say that a Hausdorff spaces X is in the class $\mathscr{C}$ whenever every continuous image of X (in particular, X itself) is a GO-space.

Main Theorem 1.1. *A Hausdorff space X belongs to the class $\mathscr{C}$ if and only if X is homeomorphic to a countable successor ordinal (with the order topology).*

These results are closely related to the following line of research: characterize Hausdorff topological spaces X such that all continuous images of X have the topological property $\mathscr{P}$. All questions listed below for concrete $\mathscr{P}$ are still open.

Problem 1.2.

(1) Characterize Hausdorff spaces such that all continuous images of X are regular.
(2) Characterize Hausdorff spaces X such that all continuous images of X are normal. (This was partially solved by W. Fleissner and R. Levy in [5,6].)
(3) Characterize Hausdorff spaces X such that all continuous images of X are monotonically normal. (This is related to "Niekel Conjecture" answered positively by M. E. Rudin [9].)
(4) Characterize Hausdorff spaces X such that all continuous images of X are realcompact. (This question has been formulated in [1].)
(5) Characterize Hausdorff spaces X such that all continuous images of X are paracompact.

Remark 1.3. Note a recent paper [11], which studies topological properties $\mathscr{P}$ that are reflectable in small continuous images. In particular the authors of [11] show that a GO-space X is hereditarily Lindelöf iff all continuous images of X have countable pseudocharacter.

Our article is organized as follows. In §2 we collect the basic facts on GO-spaces which will be used in the paper and discuss some elementary examples of spaces that do not belong to the class $\mathscr{C}$. The proof of Main Theorem 1.1 is presented in §3. First we show that any member X of $\mathscr{C}$ is a totally disconnected space, then we show that X is countably compact and paracompact, therefore X is compact. Any compact GO-space is a LOTS, hence in principle at this point our proof can be finished in view of E. K. van Douwen's theorem for orderable spaces. For the sake of completeness, we prefer to avoid using van Douwen's result so directly and conclude by several additional observations: X is a first-countable scattered compact space which ends the proof. In the last §4 we pose the new open question: characterize Hausdorff spaces X such that all *quotient* Hausdorff images of X are LOTS/GO-spaces.

In order to make this paper widely readable, we have tried to give self-contained and elementary proofs. At the same time, aiming to keep the proofs as short as possible, we explicitly use several known results from the literature.

2. Basic facts on GO-spaces

The following facts are well-known.

Proposition 2.1. *[7, 6.1] If a GO-space is either compact or connected then it is a LOTS.*

Proposition 2.2. *For any GO-space X the following holds*