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Topological properties in Whitney blocks

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ABSTRACT

Let $C(X)$ be the hyperspace of subcontinua of a continuum X. A Whitney block is a set of the form $\mu^{-1}([s,t])$, where $\mu : C(X) \to [0,1]$ is a Whitney map and $0 \leq s < t \leq 1$. In this paper, we study the following implication: if X has property P, then each Whitney block in $C(X)$ has property P. We consider the following properties: connectedness im kleinen, being absolute neighborhood retract, local contractibility, and m-mutual aposyndesis.

1. Introduction

Whitney blocks have been studied by M. E. Aguilera and A. Illanes (see [1] and [2]). To continue with this, there are some topological properties that can be considered. In Section 2, the property of connectedness im kleinen in Whitney blocks is studied.

In the case of the property of being an ANR, proposed in [1, Example 7.1], it is shown that there exists a continuum X such that there is not an ANR but all initial blocks for $C(X)$ are ANRs. In Section 3, we study the other implication.

The property of being m-mutually aposyndetic, for $m = 2$ and $m = 3$, is studied in [1, Theorem 9.2 and Theorem 9.4]. It is shown that this property, for both cases, is induced to Whitney blocks. Here, we focus mainly in the following question (see [1, Question 9.5]): Let $m \geq 4$. Is m-mutual aposyndesis induced to Whitney Blocks? We answer this question positively in the general case. The details are given in Section 4, where we introduce the notion of an m-step.

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We end this section with the following definitions. A continuum is a compact connected metric space with more than one point. Given a continuum X, a subcontinuum of X is a closed, nonempty and connected space of X. That is, $A \subset X$ is a subcontinuum of X if either A is a continuum with the induced topology or A is a one-point set. We consider the hyperspace $C(X)$ defined as

$$C(X) = \{A \subset X : A \text{ is a subcontinuum of } X\}.$$

The hyperspace $C(X)$ is considered with the Hausdorff metric H (see [3, Theorem 2.2]).

Given a continuum X with metric d, a number $\varepsilon > 0$, an element $p \in X$, and a subset A of X, let $B^d(p, \varepsilon)$ be the open ε-ball around p and $N(A, \varepsilon) = \bigcup\{B^d(a, \varepsilon) : a \in A\}$.

For a continuum X and $A, B \in C(X)$ such that $A \subset B$, an order arc from A to B is a continuous function $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$, and $\alpha(u) \subset \alpha(v)$ if $0 \le u \le v \le 1$. It is known (see [3, Theorem 14.6]) that if $A, B \in C(X)$ and $A \subset B$, then there exist order arcs from A to B.

A Whitney map for $C(X)$ is a continuous function $\mu : C(X) \to [0, 1]$ such that
(a) $\mu(\{p\}) = 0$ for each $p \in X$,
(b) if $A, B \in C(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$, and
(c) $\mu(X) = 1$.
It is known that for every continuum X, $C(X)$ admits Whitney maps (see [3, Theorem 13.4]).

A *Whitney level for* $C(X)$ is a set of the form $\mu^{-1}(t)$, where $0 < t < 1$. As it was shown in [3, Theorem 19.9], $\mu^{-1}(t)$ is a continuum for each $t \in (0, 1)$. A topological property P is said to be a *Whitney property* provided that if a continuum X has property P, so does $\mu^{-1}(t)$ for each Whitney map μ for $C(X)$ and for each $t \in [0, 1]$. See [3, Definition 27.1(a)]. Whitney properties have been widely studied. Many properties and references about Whitney levels can be found in the book [3].

A *Whitney block for* $C(X)$ is a set of the form $\mu^{-1}([s, t])$, where $0 \le s < t \le 1$. In the case that $s = 0$, they are called *initial Whitney blocks*. Each Whitney block is a subcontinuum of $C(X)$ (see [1, Theorem 2.1]). In this way, it is also natural to study Whitney blocks. In order to study Whitney blocks, the following definition is introduced (see [1]): A topological property P is considered to be *induced to Whitney blocks* provided that the following implication holds: if X has property P, then each Whitney block for $C(X)$ has property P.

2. Connectedness im kleinen

Theorem 1. *If a continuum X is connected im kleinen at some point $p \in X$, then for every Whitney map $\mu : C(X) \to [0, 1]$, for every s, t such that $0 \le s < t \le 1$ and for every element $A \in \mu^{-1}([s, t])$ such that $p \in A$, the Whitney block $\mu^{-1}([s, t])$ is connected im kleinen at A.*

Proof. Let $\mathcal{B} = \mu^{-1}([s, t])$. Consider $\mathcal{A}_p = \{A \in \mathcal{B} : p \in A\}$. Take $A \in \mathcal{A}_p$ and set $r = \mu(A)$. We check that $\mathcal{B}$ is connected im kleinen at A. Let $\varepsilon > 0$. Then there exists $\eta > 0$ such that the following implication holds: If $E, F \in C(X)$, $E \subset N(F, \eta)$ and $|\mu(F) - \mu(E)| < \eta$, then $H(E, F) < \frac{\varepsilon}{2}$. Since X is connected im kleinen at p, there exist $\delta \in (0, \eta)$ and a connected set V such that $B^d(p, \delta) \subset V \subset \mathrm{cl}_X(V) \subset B^d(p, \eta)$. Let $C_0 = \mathrm{cl}_X(V)$. Observe that $p \in A \cap C_0$, so $A \cup C_0 \in C(X)$. We can assume that $\mu(C_0) < t$. Let

$$\mathcal{V} = B^H(A, \delta) \cap \mu^{-1}((r - \eta, r + \eta)) \cap \mathcal{B}.$$

Given $B \in \mathcal{V}$, there exists $b \in B$ such that $d(p, b) < \delta$. Then $b \in B \cap B^d(p, \delta) \subset B \cap C_0$ and, as a consequence, $B \cup C_0 \in C(X)$. Therefore, $A \cup B \cup C_0 \in C(X)$. Set $E_B = A \cup B \cup C_0$. Observe that $\mu(E_B) \ge r$ and $E_B \subset A \cup N(A, \delta) \cup B^d(p, \eta) \subset N(A, \eta)$. If we take an element $E \in (\mu|_{C(E_B)})^{-1}(r)$, $E \subset N(A, \eta)$. Since $\mu(E) = r = \mu(A)$, by the election of η, we obtain that $H(A, E) < \frac{\varepsilon}{2}$.