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Dynamical characterizations of combinatorially rich sets near zero

Sourav Kanti Patra

Department of Mathematics, Ramakrishna Mission Vidyamandira, Belur Math, Howrah-711202, West Bengal, India

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Hindman and Leader first introduced the notion of Central sets near zero for dense subsemigroups of $((0, \infty), +)$ and proved a powerful combinatorial theorem about such sets. Using the algebraic structure of the Stone–Čech compactification, Bayatmanesh and Tootkabani generalized and extended this combinatorial theorem to the central sets theorem near zero. Algebraically one can define quasi-central sets near zero for dense subsemigroups of $((0, \infty), +)$, and they also satisfy the conclusion of central sets theorem near zero. In a dense subsemigroup of $((0, \infty), +)$, C-sets near zero are the sets, which satisfies the conclusions of the central sets theorem near zero. We shall produce dynamical characterizations of these combinatorially rich sets near zero.

1. Introduction

Furstenberg defined the concept of a central subset of positive integers [5, Definition 8.3] and proved several important properties of such sets using notions from topological dynamics.

Definition 1.1. A dynamical system is a pair $(X, \langle T_s \rangle_{s \in S})$ such that

(i) X is compact Hausdorff space,
(ii) S is a semigroup,

E-mail address: souravkantipatra@gmail.com.

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(iii) For each $s \in S$, $T_s : X \to X$ and T_s is continuous, and
(iv) For all s, t, $T_s \circ T_t = T_{st}$.

Inspired by the fruitful interaction between Ramsey theory and ultrafilters on semigroups, Bergelson and Hindman, with the assistance of B. Weiss, later provided an algebraic characterization of central sets of $\mathbb{N}$ in [2, Section 6]. This algebraic characterization as a definition enabled them easily to extend the notion of a central set to any semigroup.

Let us now give a brief description of the algebraic structure of βS for a discrete semigroup $(S, \cdot)$. We take the points of βS to be the ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S$. Given $A \subseteq S$ let us set, $\bar{A} = \{p \in \beta S : A \in p\}$. Then the set $\{\bar{A} : A \subseteq S\}$ is a basis for a topology on βS. The operation '$\cdot$' on S can be extended to the Stone–Čech compactification βS of S so that $(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$ the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Given $p, q \in \beta S$ and $A \in S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. A non-empty subset I of a semigroup $(T, \cdot)$ is called a left ideal of S if $T \cdot I \subseteq I$, a right ideal of S if $I \cdot T \subseteq I$ and a two sided ideal (or simply an ideal) if it is both a left and a right ideal. A minimal left ideal is a left ideal that does not contain any proper left ideal. Similarly we can define minimal right ideal and the smallest ideal. Any compact Hausdorff right topological semigroup $(T, \cdot)$ has the smallest two sided ideal

$$\begin{aligned} K(T) &= \bigcup\{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } T\}. \end{aligned}$$

We now present Bergelson's characterizations of Central sets.

Definition 1.2. Let S be a discrete semigroup and let C be a subset of S. Then C is central if and only if there is an idempotent p in $K(\beta S)$ such that $C \in p$.

To give a dynamical characterization of central sets in arbitrary semigroup S, we need the following definition and from now on, $P_f(X)$ is the set of finite non-empty subsets of X, for any set X.

Definition 1.3. ([7, Definition 3.1]) Let S be a semigroup and let $A \subseteq S$.

(a) The set A is syndetic if and only if there is some $G \in P_f(S)$ such that $S = \bigcup_{t \in G} t^{-1}A$.
(b) The set A is piecewise syndetic if and only if there is some $G \in P_f(S)$ such that for any $F \in P_f(S)$ there is some $x \in S$ with $Fx \subseteq \bigcup_{t \in G} t^{-1}A$.

Recall the definitions of proximality and uniform recurrence in a dynamical system from [3, Definition 1.2(b)] and [3, Definition 1.2(c)].

Definition 1.4. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system.

(a) A point $y \in S$ is uniformly recurrent if and only if for every neighbourhood U of y, $\{s \in S : T_s(y) \in U\}$ is syndetic.
(b) The points x and y of X are proximal if and only if for every neighbourhood U of the diagonal in $X \times X$, there is some $s \in S$ such that $(T_s(x), T_s(y)) \in U$.