Topology and its Applications 239 (2018) 152–159

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – In memory of Professor Sibe Mardešić

Characterizing slices for proper actions of locally compact groups

Sergey A. Antonyan [1]

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México, 04510 Ciudad de México, Mexico

A R T I C L E   I N F O

Article history:
Received 1 February 2017
Received in revised form 6 April 2017
Available online 6 March 2018

Dedicated to the memory of Professor Sibe Mardešić

MSC:
22F05
57S20
54C55
54H15

Keywords:
Proper action
Global slice
Orbit space
John ellipsoid

A B S T R A C T

In his seminal work [13], R. Palais extended a substantial part of the theory of compact transformation groups to the case of proper actions of locally compact groups. Here we extend to proper actions some other important results well known for compact group actions. In particular, we prove that if H is a compact subgroup of a locally compact group G and S is a small (in the sense of Palais) H-slice in a proper G-space, then the action map $G \times S \to G(S)$ is open. This is applied to prove that the slicing map $f_S : G(S) \to G/H$ is continuous and open, which provides an external characterization of a slice. Also an equivariant extension theorem is proved for proper actions. As an application, we give a short proof of the compactness of the Banach–Mazur compacta.

1. Introduction

The letter G will denote a Hausdorff topological group with unit element $e \in G$. All spaces are assumed to be completely regular and Hausdorff.

By an action of G on a space X we mean a continuous map $(g, x) \mapsto gx$ of the product $G \times X$ into X such that $ex = x$ and $(gh)x = g(hx)$, whenever $x \in X$, $g, h \in G$ and e is the unity of G. A space X together with a fixed action of the group G is called a G-space.

If X and Y are G-spaces, then a continuous map $f : X \to Y$ is called a G-map or an equivariant map, if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$. For a point $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ of G is called the *stabilizer* or *isotropy subgroup* at x. Clearly, $G_x \subset G_{f(x)}$ whenever f is a G-map and $x \in X$.

E-mail address: antonyan@unam.mx.

[1] The author was supported in part by grant IN-115717 from PAPIIT (UNAM).

https://doi.org/10.1016/j.topol.2018.02.026

If X is a G-space, then for a subset $S \subset X$ and a subgroup $H \subset G$, the H-hull (or H-saturation) of S is defined as follows: $H(S) = \{hs \mid h \in H,\ s \in S\}$. If S is the one point set $\{x\}$, then the H-hull $H(\{x\})$ usually is denoted by $H(x)$ and called the H-orbit of x. The orbit space X/H is always considered in its quotient topology. A subset $S \subset X$ is called H-invariant, if it coincides with its H-hull, i.e., $S = H(S)$. A G-invariant set is also called, simply, invariant.

For a closed subgroup $H \subset G$, by G/H we will denote the G-space of cosets $\{xH \mid x \in G\}$ under the action induced by left translations, i.e., $g(xH) = (gx)H$ whenever $g \in G$ and $xH \in G/H$.

The following well-known notion of an H-slice is the main object in this paper (cf. [12, §1.7]).

Definition 1.1. Let X be a G-space and H a closed subgroup of G. A subset $S \subset X$ is called an H-slice in X, if:

(1) S is H-invariant, i.e., $H(S) = S$,
(2) S is closed in $G(S)$,
(3) if $g \in G \setminus H$, then $gS \cap S = \emptyset$,
(4) the saturation $G(S)$ is open in X.
If in addition $G(S) = X$, then we say that S is a global H-slice of X.

One of the most fundamental facts in the theory of G-spaces when G is a compact Lie group is, the so called, Slice Theorem. In its full generality it was proved by G. Mostow [10] (see also [12, Corollary 1.7.19]).

Theorem 1.2 *(Slice Theorem). Let G be a compact Lie group and X a G-space. Then for every point $x \in X$, there exists a G_x-slice $S \subset X$ such that $x \in S$.*

To each H-slice $S \subset X$, a G-map $f_S : G(S) \to G/H$, called the *slicing map*, is associated according to the following rule:

$$f_S(gs) = gH \quad \text{for every} \quad g \in G,\ s \in S.$$

Lets check that f_S is well defined. Indeed, if $gs = g's'$ for some $g, g' \in G$ and $s, s' \in S$ then $s = g^{-1}g's' \in S \cap g^{-1}g'S$. Then item (3) of Definition 1.1 yields that $g^{-1}g' \in H$ which is equivalent to $gH = g'H$, as required. Thus, the slicing map f_S is well defined.

It is immediate that the slicing map is equivariant, i.e., $f_S(gx) = gf_S(x)$ for all $x \in G(S)$ and $g \in G$. It is also clear that $S = f_S^{-1}(eH)$, where $eH \in G/H$ stands for the coset of the unit element $e \in G$.

It is a well known fact that the slicing map is continuous whenever the acting group G is compact (see e.g., [12, Theorem 1.7.7]); this gives the following important external characterization of an H-slice.

Theorem 1.3. *Let G be a compact group, H a closed subgroup of G, and X a G-space. Then there exists a one-to-one correspondence between all equivariant maps $f : X \to G/H$ and global H-slices S in X given by $f \mapsto S_f := f^{-1}(eH)$. The inverse correspondence is given by $S \mapsto f_S$, above defined.*

Below, in Theorem 2.2 we generalize Theorem 1.3 to the case of proper actions of arbitrary locally compact groups, which are an important generalization of actions of compact groups. This result is proceeded by Theorem 2.1 which establishes some important properties of small global slices. Then these results are applied in Section 3 to orbit spaces of proper G-spaces. In Section 4 we prove an equivariant extension theorem which is further applied to get an equivariant extension of a continuous map defined on a small cross section. In the final Section 5 the results of Sections 2 and 3 are applied to give a short proof of the compactness of the Banach–Mazur compacta $BM(n)$, $n \geq 1$.