Topology and its Applications 239 (2018) 234–243

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – In memory of Professor Sibe Mardešić

Some anomalous examples of lifting spaces

Gregory R. Conner [a,*,1], Wolfgang Herfort [b], Petar Pavešić [c,2]

[a] Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
[b] Institute for Analysis and Scientific Computation, Technische Universität Wien, Wiedner Hauptstraße 8-10/101, Vienna, Austria
[c] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 21 Ljubljana, Slovenia

ARTICLE INFO

Article history:
Received 22 December 2016
Available online 6 March 2018

To the memory of Sibe Mardešić

MSC:
54B25
54B35
54H20
54C

Keywords:
Fibrations
Topological dynamics
Morse–Thue system
Inverse spectra

ABSTRACT

An inverse limit of a sequence of covering spaces over a given space X is not, in general, a covering space over X but is still a *lifting space*, i.e. a Hurewicz fibration with unique path lifting property. Of particular interest are inverse limits of finite coverings (resp. finite regular coverings), which yield fibrations whose fibre is homeomorphic to the Cantor set (resp. profinite topological group). To illustrate the breadth of the theory, we present in this note some curious examples of lifting spaces that cannot be obtained as inverse limits of covering spaces.
© 2018 Elsevier B.V. All rights reserved.

1. Introduction

For any continuous map $p\colon L \to X$ let $p_*\colon L^I \to X^I$ be the induced map between the respective spaces of continuous paths: $p_*\colon (\alpha\colon I \to L) \mapsto (p \circ \alpha\colon I \to X)$.

A map $p\colon L \to X$ is said to be a *lifting projection* if the following diagram

$$\begin{tikzcd} L^I \arrow[r, "\mathrm{ev}_0"] \arrow[d, "p_*"'] & L \arrow[d, "p"] \\ X^I \arrow[r, "\mathrm{ev}_0"'] & X \end{tikzcd}$$

---

* Corresponding author.
E-mail addresses: conner@mathematics.byu.edu (G.R. Conner), wolfgang.herfort@tuwien.ac.at (W. Herfort), petar.pavesic@fmf.uni-lj.si (P. Pavešić).
[1] The first author is supported by Simons Foundation collaboration grant 246221.
[2] The third author was supported by the Slovenian Research Agency research grant P1-0292 and research project J1-7025.

https://doi.org/10.1016/j.topol.2018.02.027
0166-8641/© 2018 Elsevier B.V. All rights reserved.

is a pull-back in the category of topological spaces. In other words, there is a natural continuous one-to-one correspondence between paths in L, and pairs (α, l), where α is a path in X and $l \in L$ with $p(l) = \alpha(0)$. In fact, by [7, Theorem 1.1] a lifting projection $p\colon L \to X$ is automatically a Hurewicz fibration. Moreover, given a path $\alpha\colon I \to X$ and a point $l \in L$, such that $p(l) = \alpha(0)$ there exists a unique path $\widetilde{\alpha}\colon I \to L$, such that $\widetilde{\alpha}(0) = l$ and $p \circ \widetilde{\alpha} = \alpha$. In particular, every covering projection $p\colon \widetilde{X} \to X$ is a lifting projection.

A *lifting space* is a triple (L, p, X) where $p\colon L \to X$ is a lifting projection. As usual, X is the *base* and L is the *total space* of the fibration, but we will occasionally abuse the terminology and refer to the space L as a lifting space over X. For every $x \in X$ the preimage $p^{-1}(x) \subset L$ is the *fibre* of p over x. If X is path-connected, then all fibres are homeomorphic, so we can speak about *the* fibre of p.

In his classical textbook on algebraic topology E. Spanier [8, Chapter II] develops most of the theory of covering spaces within the general framework of lifting spaces (which he calls *fibrations with unique path-lifting property*). As we already mentioned, covering spaces are prime examples of lifting spaces. Conversely, if the base space X is sufficiently "nice" (i.e. locally path-connected and semi-locally simply connected), then the coverings over X are exactly the lifting spaces over X with a path-connected and locally path-connected total space ([8, Theorem 2.4.10]). However, if we allow more general base spaces or total spaces that are connected but non necessarily path-connected, then a whole new world arises.

Indeed, many authors have studied unusual and pathological examples of covering spaces and tried to build a suitable general theory. One should mention in particular Fox's theory of overlays [5], an interesting generalization of the concept of coverings by Fischer and Zastrow [6], and a theory of coverings specially geared toward locally path-connected spaces by Brodsky et al. [1]. See also Dydak's short note [4] that was written very much in the spirit of the present article. However, our approach is different in that we do not attempt to simply extend the concept of a covering space to more general bases but rather we pursue the theory of lifting spaces in the sense of Spanier. It includes covering spaces as a special case, but it has a richer structure even when the base is very simple, like a circle or a cell-complex.

Surprisingly, it turns out that despite the generality, the theory of lifting spaces has many pleasant properties that are not shared by covering spaces. Most notably, lifting spaces are preserved by arbitrary products, compositions and inverse limits, which is not the case for covering spaces. While a general exposition on lifting spaces can be found in [3] and a classification of compact lifting spaces is in preparation, in this note we describe some curious examples of lifting spaces that somehow defy classification and reveal interesting connections with dynamical systems and profinite groups.

2. Examples of lifting spaces

As explained in the Introduction, in order to find lifting spaces that are not already covering spaces we must relax some of the usual assumptions either about the base or about the total space. We will mostly work with very simple base spaces (the circle and the figure eight-space) and concentrate on the intricacies of the topology of the total space. In particular, we will assume that the total spaces are connected, but are not necessarily path-connected or locally path-connected. Only in our final example we will describe an interesting lifting space over the Hawaiian earring.

It is actually not difficult to construct lifting spaces that are not covering spaces as we have the following characterization.

Proposition 2.1. *([8, Theorem II.2.5]) A map $p\colon L \to X$ is a lifting space if, and only if it is a Hurewicz fibration with totally path-disconnected fibre.*

Furthermore, a locally trivial projection over a paracompact space is always a Hurewicz fibration (see [7, Theorem 1.6, Corollary 1.7]). Conversely, if $p\colon L \to X$ is a lifting space, then the restriction of p over a contractible subspace of X is locally trivial.

Download English Version:

https://daneshyari.com/en/article/8904104

Download Persian Version:

https://daneshyari.com/article/8904104

Daneshyari.com