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Cobordism classes of maps and covers for spheres

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ABSTRACT

In this paper we show that for $m > n$ the set of cobordism classes of maps from m-sphere to n-sphere is trivial. The determination of the cobordism homotopy groups of spheres admits applications to the covers for spheres.

1. Introduction

Let M_1 and M_2 be compact oriented manifolds of dimension m. Two continuous maps $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$ are called cobordant if there are a compact oriented manifold W with $\partial W = M_1 \sqcup M_2$ and a continuous map $F : W \to X$ such that $F|_{M_i} = f_i$ for $i = 1, 2$. Note that the set of cobordism classes $f : S^m \to X$ form a group $\pi_m^C(X)$ that is a quotient of $\pi_m(X)$.

In Section 2 we consider assumptions for X such that $\pi_m^C(X) = 0$ (Theorem 2.1). In particular, Corollary 2.2 states that $\pi_n^C(S^n) = \pi_n(S^n) = \mathbb{Z}$ and if $m > n$ then

$$\pi_m^C(S^n) = 0.$$

In Section 3 we show that for manifolds the homotopy and cobordism classes of covers are equivalent to the homotopy and cobordism classes of their associated maps. Then we can apply results of Sections 2 for covers, in particular, see Corollary 3.6.

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2. Cobordism classes of maps for spheres

Consider a group of oriented cobordism classes of maps $\Omega_*^{SO}(X)$ [1, Chapter 1]. Let M_i, $i = 1, 2$, be compact oriented manifolds without boundary of dimension m. Let $f_i : M_i \to X$, $i = 1, 2$, be continuous maps to a space X. Then $[f_1]_C = [f_2]_C$ in $\Omega_m^{SO}(X)$, i.e. maps f_i are *cobordant* if there are a compact oriented manifold W with $\partial W = M_1 \sqcup M_2$ and a continuous map $F : W \to X$ such that $F|_{M_i} = f_i$ for $i = 1, 2$.

If $M_2 = \emptyset$, then $[f_1]_C = 0$. In this case f_1 is called *null-cobordant*.

Let M be a compact oriented manifold without boundary. We denote the set of cobordism classes of $f : M \to X$ by $[M, X]_C$.

Theorem 2.1. *Let X be a finite CW-complex whose integral homology $H_*(X, \mathbb{Z})$ has only 2-torsion. Let $f : S^m \to X$ be a map that induces zero homomorphism of m-dimensional cohomology with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$. Then f is null-cobordant in $\Omega_m^{SO}(X)$ the image of f is 0. In particular, $[S^m, X]_C = 0$ if $\dim X < m$.*

Proof. By [1, Theorem 17.6], the cobordism class of $f : S^m \to X$ is determined by the Pontrjagin numbers and the Stiefel–Whitney numbers of the map f. From the definition, the Pontrjagin numbers and the Stiefel–Whitney numbers of the map f are determined by its induced homomorphisms on cohomology with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$, respectively. The hypothesis in the statement guarantees that f and the constant map induce the same homomorphism on cohomology with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$, and hence the result. □

Let M be an m-dimensional sphere S^m. In this case denote $[M, S^n]_C$ by $\pi_m^C(S^n)$. It is easy to prove that the cobordism classes $\pi_m^C(S^n)$ form a group. Moreover, there is a subgroup N in $\pi_m(S^n)$ such that

$$\pi_m^C(S^n) = \pi_m(S^n)/N.$$

Corollary 2.2. *If $m \neq n$, then $\pi_m^C(S^n) = 0$, otherwise $\pi_n^C(S^m) = \mathbb{Z}$.*

Proof. We obviously have the case $m < n$. Theorem 2.1 yields the most complicated case.

Let $m = n$. The Hopf degree theorem (see [4, Sect. 7]) states that two continuous maps $f_1, f_2 : S^n \to S^n$ are homotopic, i.e. $[f_1] = [f_2]$ in $\pi_n(S^n)$, if and only if $\deg f_1 = \deg f_2$. It is clear that $[f_1] = [f_2]$ implies $[f_1]_C = [f_2]_C$. Now we show that from $[f_1]_C = [f_2]_C$ follows $\deg f_1 = \deg f_2$. Indeed, then we have $F : W \to S^n$ with $F|_{M_i = S^n} = f_i$. Note that $Z := F^{-1}(x)$ for a regular $x \in S^n$, is a manifold of dimension one. It is easy to see that a cobordism (Z, Z_1, Z_2), where $Z_i := f_i^{-1}(x)$, implies $\deg f_1 = \deg f_2$. Thus, $\pi_n^C(S^n) = \pi_n(S^n) = \mathbb{Z}$. □

Corollary 2.2 states that $f : S^m \to S^n$ is null-cobordant for $m > n$. Therefore, we have the following result.

Corollary 2.3. *Let $m > n$. Then for any continuous map $f : S^m \to S^n$ there are a compact oriented manifold W with $\partial W = S^m$ and a continuous map $F : W \to S^n$ such that F on the boundary coincides with f.*

Remark. In the earlier version of this paper, we had a proof that $\pi_m^C(S^n) = 0$, where $m > n$ only for particular cases. We formulated this statement as a conjecture and sent the preprint to several topologists. Soon, Diarmuid Crowley sent us a sketch of the proof of this conjecture. Later, Alexey Volovikov pointed out to us that Theorem 2.1 follows easily from [1, Theorem 17.6].