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On loose Legendrian knots in rational homology spheres

Alberto Cavallo

Department of Mathematics and its Application, Central European University, Budapest 1051, Hungary

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ABSTRACT

We prove that loose Legendrian knots in a rational homology contact 3-sphere, satisfying some additional hypothesis, are Legendrian isotopic if and only if they have the same classical invariants. The proof requires a result of Dymara on loose Legendrian knots and Eliashberg's classification of overtwisted contact structures on 3-manifolds.

1. Introduction

Knot theory in contact 3-manifolds turned out to be a very interesting field to study. In this setting, an oriented knot is called Legendrian if it is everywhere tangent to the contact structure and two such knots are said to be equivalent if they are Legendrian isotopic; that is, there is an isotopy between them such that they are Legendrian at any step. In the last twenty years much work has been done in order to find some criteria to determine whether two Legendrian knots are Legendrian isotopic or not. Three invariants can be immediately defined from the definition of Legendrian knot. For this reason they are usually called classical invariants.

The first one is the knot type, that is the smooth isotopy class of our oriented Legendrian knot K. The knot type of K is a Legendrian invariant; in fact it is known that two Legendrian knots are Legendrian isotopic if and only if there is an ambient contact isotopy of the 3-manifold sending the first knot into the second one as shown in [7].

The other two classical invariants are the Thurston–Bennequin number, which is defined as the linking number of the contact framing of K respect to a Seifert framing of K, and the rotation number; the latter

E-mail address: cavallo_alberto@phd.ceu.edu.

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being the numerical obstruction to extending a non-zero vector field, everywhere tangent to the knot, to a Seifert surface of K (see [7]). These two invariants are usually well-defined only for null-homologous knots in a rational homology 3-sphere, but a generalization exists for every Legendrian knot in such a manifold. See [1] for details.

Legendrian knots in overtwisted contact 3-manifolds come in two types: loose and non-loose. A Legendrian knot is loose if also its complement is overtwisted, while it is non-loose if the complement is tight. More explicitly, a Legendrian knot is loose if and only if we can find an overtwisted disk that is disjoint from the knot. While it was known that non-loose Legendrian knots are not classified by their classical invariants [11], in the case of loose knots such example was found only recently by Vogel [12]; even though, according to [6], this phenomenon was already known to Chekanov. Conversely, there were some results that go in the opposite direction.

Etnyre's coarse classification of loose Legendrian knots [9] is probably the most important one. It says that loose knots are completely determined by their classical invariants, but only up to contactomorphism, which is a weaker relation than Legendrian isotopy. Another result was proved by Dymara in [3] and it states that two Legendrian knots, with the same classical invariants, such that the complement of their union contains an overtwisted disk are Legendrian isotopic (Theorem 2.4). This result holds only in rational homology spheres.

In this paper we show that Dymara's result can be strengthened. In fact we prove the following theorem.

Theorem 1.1. *Consider a rational homology contact 3-sphere (M, ξ). Suppose that there are two loose Legendrian knots K_1 and K_2 in (M, ξ) such that there exists a pair of disjoint overtwisted disks (E_1, E_2), where E_i is contained in the complement of K_i for $i = 1, 2$. Then K_1 and K_2 are Legendrian isotopic if and only if they have the same classical invariants.*

Though we still need an assumption on the overtwisted disks, this version can be applied in many interesting cases like disjoint unions of Legendrian knots. We say that a Legendrian 2-component link L is split if (M, ξ) can be decomposed into $(M_1 \# M_2, \xi_1 \# \xi_2)$ and $K_i \hookrightarrow (M_i, \xi_i)$ for $i = 1, 2$. In other words, if L is the disjoint union of K_1 and K_2.

Corollary 1.2. *Suppose K_1 and K_2 are two loose Legendrian knots in the rational homology contact sphere (M, ξ) such that $K_1 \cup K_2$ is a split Legendrian link. Then they are Legendrian isotopic if and only if they have the same classical invariants.*

This paper is organized as follows. In Section 2 we define connected sums for contact 3-manifolds and Legendrian links and we prove Theorem 1.1. In Section 3 we explain what is the disjoint union of Legendrian knots and give a precise definition of split Legendrian links. Moreover, we apply our main result to this kind of loose knots.

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2. A classification theorem for loose Legendrian knots

2.1. Contact and Legendrian connected sum

The definition of the connected sum of two 3-manifolds can be easily given also in the contact setting. Let us take two connected contact manifolds (M_1, ξ_1) and (M_2, ξ_2); we call M_i' (for $i = 1, 2$) the manifolds