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# Dually properties and cardinal inequalities ☆

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ABSTRACT

In this paper, we prove that DCCC is self-dual with respect to neighborhood assignments. Some related results on dually CCC (or dually DCCC) spaces are also obtained. Moreover, we prove that the cardinality of a dually CCC space $X$ is at most $2^\omega$ if $X$ satisfies one of the following conditions: (1) $X$ has a rank 2-diagonal; (2) $X$ is first countable and has a $G_\delta$-diagonal; (3) $X$ is first countable and perfect. Finally, we prove that the extent of a first countable dually CCC space is at most $2^\omega$.

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## 1. Introduction

The notation of dually property was introduced by J. van Mill et al. in [9] and studied in [2], [5] and [3], which is a development of an idea of E. van Douwen used to define $D$-spaces. A *neighborhood assignment* for a space $(X,\tau)$ is a function $\phi : X \to \tau$ with $x \in \phi(x)$ for every $x \in X$. A set $Y \subset X$ is a *kernel* of $\phi$ if $\phi(Y) = \{\phi(y) : y \in Y\}$ covers $X$. Given a property (or a class) $\mathcal{P}$, the class $\mathcal{P}^*$ dual to $\mathcal{P}$ (with respect to neighborhood assignments) consists of spaces $X$ such that for any neighborhood assignment $\phi$ on $X$, there is $Y \subset X$ with property $\mathcal{P}$ and $\phi(Y) = \{\phi(y) : y \in Y\}$ covers $X$. The spaces from $\mathcal{P}^*$ are called *dually* $\mathcal{P}$. Let us recall some other definitions.

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**Definition 1.1.** A space $X$ has *countable chain condition* (CCC for short) if every disjoint family of non-empty open sets of $X$ is countable.

**Definition 1.2.** A space $X$ has *discrete countable chain condition* (DCCC for short) if every discrete family of non-empty open sets of $X$ is countable.

In Section 3, we prove that DCCC is self-dual with respect to neighborhood assignments (Proposition 3.1). By using this result, we show that if $\mathcal{P}$ is a property (or a class) lying between CCC and DCCC, then in the class of perfectly normal spaces $\mathcal{P}$ is self-dual with respect to neighborhood assignments (Proposition 3.3). We also prove that if $X$ is a weakly $\omega_1$-collectionwise Hausdorff and dually CCC space then $e(X) \leq \omega$ (Proposition 3.6). In Section 4, we prove that the cardinality of a dually CCC space $X$ is at most $2^\omega$ if $X$ satisfies one of the following conditions: (1) $X$ has a rank 2-diagonal (Theorem 4.5); (2) $X$ is first countable and has a $G_\delta$-diagonal (Theorem 4.8); (3) $X$ is first countable and perfect (Proposition 4.7). Finally, we prove that the extent of a first countable dually CCC space is at most $2^\omega$ (Theorem 4.11).

## 2. Notation and terminology

All spaces are assumed to be Hausdorff unless otherwise stated.

We write $\omega$ for the first infinite cardinal, $\omega_1$ the first uncountable cardinal and $2^\omega$ for the cardinality of the continuum.

**Definition 2.1.** A space $X$ is said to be *weakly Lindelöf* if every open cover $\mathcal{U}$ of $X$ contains a countable subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup \mathcal{V}$ is dense in $X$.

Clearly, the property of being weakly Lindelöf lies between CCC and DCCC.

**Example 2.2.** (1) The space $\omega_1$ with the order topology is countably compact, so it is DCCC. But the open cover $\mathcal{U} = \{[0, \alpha] : \alpha < \omega_1\}$ of $\omega_1$ witnesses that $\omega_1$ is not weakly Lindelöf. (2) The space $(\omega_1 + 1)$ with the order topology is compact (and hence, weakly Lindelöf), which is not CCC.

**Definition 2.3.** The *extent* of a space $X$, denoted by $e(X)$, is the supremum of the cardinalities of closed discrete subsets of $X$.

**Definition 2.4.** A space $X$ is called *perfect* if every closed subset of $X$ is a $G_\delta$-set.

Note that a space is called perfectly normal if and only if it is normal and perfect.

**Definition 2.5.** If $X$ is a space and $A \subset X$, say that a family $\mathcal{U}$ is an *open expansion* of $A$ if $\mathcal{U} = \{U_a : a \in A\}$ and $U_a \in \tau(a, X)$ for any $a \in A$.

**Definition 2.6.** A space $X$ is *weakly $\omega_1$-collectionwise Hausdorff* if for any closed discrete set $D \subset X$ with $|D| = \omega_1$ there exists a set $E \subset D$ such that $|E| = \omega_1$ and $E$ has a disjoint open expansion.

All notations and terminology not explained here are given in [6].

## 3. Dually DCCC and dually CCC

It was shown in [5] that there exist dually separable spaces which are not weakly Lindelöf. Therefore separability, CCC and weak Lindelöfness are not self-dual with respect to neighborhood assignments. But in

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