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# First order theory of cyclically ordered groups

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This paper is dedicated to François Lucas who died on April 2016.

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By a result known as Rieger's theorem (1956), there is a one-to-one correspondence, assigning to each cyclically ordered group $H$ a pair $(G, z)$ where $G$ is a totally ordered group and $z$ is an element in the center of $G$, generating a cofinal subgroup $\langle z\rangle$ of $G$, and such that the cyclically ordered quotient group $G/\langle z\rangle$ is isomorphic to $H$. We first establish that, in this correspondence, the first-order theory of the cyclically ordered group $H$ is uniquely determined by the first-order theory of the pair $(G, z)$. Then we prove that the class of cyclically orderable groups is an elementary class and give an axiom system for it. Finally we show that, in contrast to the fact that all theories of totally ordered Abelian groups have the same universal part, there are uncountably many universal theories of Abelian cyclically ordered groups.



## 1. Introduction and basic facts

The study of cyclically ordered groups (c.o. groups) was initiated in [14]. Definitions and notations not given here, about c.o. group and totally ordered groups (t.o. groups, or linearly ordered groups), can be found in [4] (IV, 6, pp. 61–65), [8], [18] and [16]. The terminology from model theory can be found in [1].

We say that $(A, R)$ is a cyclically ordered set (or $R$ is a cyclic order on $A$) if $A$ is a set and $R$ is a ternary relation on $A$ satisfying the following axioms $R_1$ to $R_5$:

$R_1$: $\forall x, y, z\ (R(x, y, z) \Rightarrow x \neq y \neq z \neq x)$, ($R$ is strict);

$R_2$: $\forall x, y, z\ (R(x, y, z) \Rightarrow R(y, z, x))$, ($R$ is cyclic);

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$R_3$: $\forall x, y, z\ (x \neq y \neq z \neq x \Rightarrow (R(x, y, z) \vee R(x, z, y)))$, ($R$ is total);

$R_4$: $\forall x, y, z\ (R(x, y, z) \Rightarrow \neg R(x, z, y)$, ($R$ is antisymmetric);

$R_5$: $\forall x, y, z, u\ (R(x, y, z) \wedge R(y, u, z) \Rightarrow R(x, u, z))$, ($R$ is transitive).

Note that $x$ being fixed, we deduce from $R_3$, $R_4$ and $R_5$ that $R(x, \cdot, \cdot)$ induces a linear order on the set $A \backslash \{x\}$.

A simple example of a cyclically ordered set is given by a circle $C$ that one traverses clockwise (or counterclockwise). Assume that one starts from a point of $C$ and the points $x$, $y$, $z$ are found in this order, then we set $R(x, y, z)$. Starting from another point, one can also find the same points in the order $y$, $z$, $x$ or $z$, $x$, $y$, so we have also $R(y, z, x)$ and $R(z, x, y)$. One sees that the other properties hold.

We say that $(G, R)$ (or in short $G$) is a cyclically ordered group (c.o. group) if $R$ is a cyclic order on the underlying set of $G$ which is compatible with the group operation of $G$, i.e. it satisfies:

$R_6$: $\forall x, y, z, u, v\ (R(x, y, z) \Rightarrow R(uxv, uyv, uzv))$ ($R$ is compatible).

For example, the group of complex numbers of norm 1 can be seen as the unit circle traversed counterclockwise, and it is a cyclically ordered group (see 1.2.1).

One can check that in a c.o. group with unit $e$, $R(e, x, y)$ implies $R(e, y^{-1}, x^{-1})$. (Note that $R$ is determined by its projection: $\{(x, y);\ R(e, x, y)\}$.)

We shall often let $R(x_1, x_2, \ldots, x_n)$ stand for $R(x_1, x_2, x_3), R(x_1, x_3, x_4), \ldots, R(x_1, x_{n-1}, x_n)$. This is equivalent to saying that for every triple $(i, j, k)$ with $1 \leq i < j < k \leq n$ we have: $R(x_i, x_j, x_k)$. In the unit circle, $R(x_1, \ldots, x_n)$ means that $x_1, \ldots, x_n$ are found in this order.

The language of c.o. groups will be here $L_c = \{\cdot, R, e, ^{-1}\}$, where the first predicate stands for the group operation, $R$ for the ternary relation, $e$ for the group identity and $^{-1}$ for the inverse function. (When considering Abelian c.o. groups we shall also use the usual symbols $+, 0, -$.) Note that the theory of cyclically ordered groups has a finite set of universal axioms in $L_c$.

If $G$ is a c.o. group, $H$ is a normal subgroup of $G$, and $g \in G$, then we shall let $\overline{g}$ stand for $gH$ whenever it yields no ambiguity.

The positive cone of $G$ is the set $P(G) = P = \{x;\ R(e, x, x^2)\} \cup \{e\}$ (see [18], p. 547, or [4]). Clearly $P \cap P^{-1} = \{e\}$ and $G = P \cup P^{-1} \cup \{x;\ x^2 = e\}$. We shall set $|x| = x$ if $x \in P$ and $|x| = x^{-1}$ if $x \notin P$. Note that the positive cone $P$ of a c.o. group does not always satisfy $P \cdot P \subseteq P$ (for example in the (additive) c.o. group $\mathbb{Z}/3\mathbb{Z}$ where $R(\overline{0}, \overline{1}, \overline{2})$ and $\mathbb{Z}$ is the additive group of integers, we have $\overline{1} \in P$ and $\overline{1} + \overline{1} \notin P$).

If $G$ is a group, then $Z(G)$ will denote its center. If $g \in G$ (or $H \subseteq G$ respectively), then $g$ (resp. $H$) is said to be central if $g$ (resp. $H$) lies in the center of $G$: $g \in Z(G)$ (resp. $H \subseteq Z(G)$). If $G$ is a t.o. group, then $g$ (resp. $H$) is said to be cofinal in $G$ if the subgroup $\langle g \rangle$ generated by $g$ (or $\langle H \rangle$ generated by $H$ respectively) is cofinal in $G$. In particular, $g > e$ is cofinal if for every $x > e$ there exist $n$ in the set $\mathbb{N}^*$ of positive integers such that $ng > x$.

A c-homomorphism from $(G, R)$ to $(G', R')$ is a group homomorphism $f$ such that for every $x_1$, $x_2$, $x_3$ in $G$, if $R(x_1, x_2, x_3)$ holds and $f(x_1) \neq f(x_2) \neq f(x_3) \neq f(x_1)$, then $R'(f(x_1), f(x_2), f(x_3))$ holds.

We will need the following fundamental constructions 1.1, 1.2, 1.3.

### *1.1. Linear cyclically ordered groups*

A t.o. group $G$ is cyclically ordered by the relation given by: $R(x, y, z)$ if, and only if, $(x < y < z \vee y < z < x \vee z < x < y)$. In this case $G$ is said to be the cyclically ordered group associated with $(G, \leq)$ and that $G$ is a linear c.o. group. (Obviously a c.o. group $G$ is linear if, and only if, $P \cdot P \subset P$.) We have $e < x$ if, and only if, $R(e, x, x^2)$. This in turn holds if, and only if, $|x| = x$ and $x \neq 0$ (in this case $|x|$ has the same meaning in the linear c.o. group as it usually has in the t.o. group).

J. Jakubík and C. Pringerová proved (see [8], Lemma 3) that a c.o. group $G$ is a linear c.o. group if, and only if, it satisfies the following system $\{\alpha\} \cup \{\beta_n\}_{n \in \mathbb{N};\ n > 1}$ of axioms:

$\alpha$: $\forall x\ (x \neq e \Rightarrow x^2 \neq e)$

$\beta_n$: $\forall x\ R(e, x, x^2) \Rightarrow R(e, x, x^n)$.