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Fixed points of self-embeddings of models of arithmetic

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We investigate the structure of *fixed point sets* of self-embeddings of models of arithmetic. Our principal results are Theorems A, B, and C below.
In what follows $\mathcal{M}$ is a countable nonstandard model of the fragment $\mathrm{I}\Sigma_1$ of PA (Peano Arithmetic); $\mathbb{N}$ is the initial segment of $\mathcal{M}$ consisting of standard numbers of $\mathcal{M}$; $\mathrm{I}_{\mathrm{fix}}(j)$ is the longest initial segment of fixed points of j; $\mathrm{Fix}(j)$ is the fixed point set of j; $K^1(\mathcal{M})$ consists of Σ_1-definable elements of $\mathcal{M}$; and a self-embedding j of $\mathcal{M}$ is said to be a proper initial self-embedding if $j(\mathcal{M})$ is a proper initial segment of $\mathcal{M}$.

Theorem A. *The following are equivalent for a proper initial segment I of $\mathcal{M}$:*

(1) $I = \mathrm{I}_{\mathrm{fix}}(j)$ *for some self-embedding j of $\mathcal{M}$.*
(2) *I is closed under exponentiation.*
(3) $I = \mathrm{I}_{\mathrm{fix}}(j)$ *for some proper initial self-embedding j of $\mathcal{M}$.*

Theorem B. *The following are equivalent for a proper initial segment I of $\mathcal{M}$:*

(1) $I = \mathrm{Fix}(j)$ *for some self-embedding j of $\mathcal{M}$.*
(2) *I is a strong cut of $\mathcal{M}$ and $I \prec_{\Sigma_1} \mathcal{M}$.*
(3) $I = \mathrm{Fix}(j)$ *for some proper initial self-embedding j of $\mathcal{M}$.*

Theorem C. *The following are equivalent:*

(1) $\mathrm{Fix}(j) = K^1(\mathcal{M})$ *for some self-embedding j of $\mathcal{M}$.*
(2) *$\mathbb{N}$ is a strong cut of $\mathcal{M}$.*
(3) $\mathrm{Fix}(j) = K^1(\mathcal{M})$ *for some proper initial self-embedding j of $\mathcal{M}$.*

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1. Introduction

In the early 1970s Harvey Friedman [10, Thm. 4.4] proved a remarkable theorem: Every countable nonstandard model $\mathcal{M}$ of PA carries a proper initial self-embedding j; i.e., j isomorphically maps $\mathcal{M}$ onto a proper initial segment of $\mathcal{M}$. Friedman's theorem has been generalized and refined in several ways over the past several decades (most recently in [26] and [9]). In the mid-1980s Ressayre [19], and independently Dimitracopoulos & Paris [4], generalized Friedman's theorem by weakening PA to the fragment $\mathrm{I}\Sigma_1$ of PA. In this paper we refine their work by investigating *fixed point sets* of self-embeddings of countable nonstandard models of $\mathrm{I}\Sigma_1$.

Our work here was inspired by certain striking results concerning the structure of fixed point sets of *automorphisms of countable recursively saturated models of* PA summarized in Theorem 1.1 below. In what follows $\mathbb{N}$ is the initial segment of $\mathcal{M}$ consisting of the standard numbers of $\mathcal{M}$; $K(\mathcal{M})$ is the set of definable elements of $\mathcal{M}$; $\mathrm{I}_{\mathrm{fix}}(j)$ is the longest initial segment of fixed points of j; and $\mathrm{Fix}(j)$ is the fixed point set of j, in other words:

$$\mathrm{I}_{\mathrm{fix}}(j) := \{m \in M : \forall x \leq m\ j(x) = x\}, \quad \text{and} \quad \mathrm{Fix}(j) := \{m \in M : j(m) = m\}.$$

Theorem 1.1. *Suppose $\mathcal{M}$ is a countable recursively saturated model of* PA*, and I is a proper initial segment of $\mathcal{M}$.*

- **(a)** (Smoryński [23]) *$I = \mathrm{I}_{\mathrm{fix}}(j)$ for some automorphism j of $\mathcal{M}$ iff I is closed under exponentiation.*[1]
- **(b)** (Kaye–Kossak–Kotlarski [15]) *$I = \mathrm{Fix}(j)$ for some automorphism j of $\mathcal{M}$ iff (I is a strong cut of $\mathcal{M}$ and $I \prec \mathcal{M}$).*
- **(c)** (Kaye–Kossak–Kotlarski [15]) *$\mathrm{Fix}(j) = K(\mathcal{M})$ for some automorphism j of $\mathcal{M}$ iff $\mathbb{N}$ is a strong cut of $\mathcal{M}$.*[2]

In this paper we formulate and establish appropriate analogues of each part of Theorem 1.1 for *self-embeddings of countable nonstandard models of* $\mathrm{I}\Sigma_1$, as encapsulated in Theorem 1.2 below. In part (c), $K^1(\mathcal{M})$ consists of Σ_1-definable elements of $\mathcal{M}$.

Theorem 1.2. *Suppose $\mathcal{M}$ is a countable nonstandard model of $\mathrm{I}\Sigma_1$, and I is a proper initial segment of $\mathcal{M}$.*

- **(a)** *$I = \mathrm{I}_{\mathrm{fix}}(j)$ for some self-embedding j of $\mathcal{M}$ iff I is closed under exponentiation iff $I = \mathrm{I}_{\mathrm{fix}}(j)$ for some proper initial self-embedding j of $\mathcal{M}$.*
- **(b)** *$I = \mathrm{Fix}(j)$ for some self-embedding j of $\mathcal{M}$ iff (I is a strong cut of $\mathcal{M}$ and $I \prec_{\Sigma_1} \mathcal{M}$) iff $I = \mathrm{Fix}(j)$ for some proper initial self-embedding j of $\mathcal{M}$.*
- **(c)** *$\mathrm{Fix}(j) = K^1(\mathcal{M})$ for some self-embedding j of $\mathcal{M}$ iff $\mathbb{N}$ is a strong cut in $\mathcal{M}$ iff $\mathrm{Fix}(j) = K^1(\mathcal{M})$ for some proper initial self-embedding j of $\mathcal{M}$.*

The plan of the paper is as follows: Section 2 reviews preliminaries; Section 3 establishes some useful basic results about self-embeddings; and Sections 4, 5, and 6 are respectively devoted to the proofs of parts (a), (b), and (c) of Theorem 1.2. Some further results and open questions are presented in Section 7.

[1] Smoryński established the right-to-left direction of this result and left the status of the other, much easier direction as an open problem. It is unclear who first established the easier direction, but by now it is considered part of the folklore of the subject. A different proof of (a stronger version of) Smoryński's theorem was established in [6].

[2] This result was generalized in [7] by showing that if $\mathbb{N}$ is strong in $\mathcal{M}$, then the isomorphism types of fixed point sets of automorphisms of $\mathcal{M}$ are precisely the isomorphism types of elementary submodels of $\mathcal{M}$, thus confirming a conjecture of Schmerl.

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