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Elementary equivalence of rings with finitely generated additive groups

Alexei G. Myasnikov [a], Francis Oger [b,*], Mahmood Sohrabi [a,*]

[a] *Department of Mathematical Sciences, Stevens Institute of Technology, Castle Point on Hudson, Hoboken, NJ 07030, USA*
[b] *UFR de Mathématiques, Université Paris 7, Bâtiment Sophie Germain, 8 place Aurélie Nemours, 75013 Paris, France*

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We give algebraic characterizations of elementary equivalence between rings with finitely generated additive groups. They are similar to those previously obtained for finitely generated nilpotent groups. Here, the rings are not supposed associative, commutative or unitary.

1. Main results

We call a *ring* any structure $(R, +, .)$ such that $(R, +)$ is an abelian group and $(x, y) \to x.y$ is bilinear. We say that it is a *scalar ring* if it is commutative, associative and unitary. We denote by $\mathbb{N}^*$ the set of strictly positive integers.

The reader is referred to [2] for the notions of mathematical logic which are used here. For each language $\mathcal{L}$, two $\mathcal{L}$-structures M, N are said to be *elementarily equivalent* if they satisfy the same first-order sentences.

* Corresponding authors.
E-mail addresses: amiasnikov@gmail.com (A.G. Myasnikov), oger@math.univ-paris-diderot.fr (F. Oger), msohrab1@stevens.edu (M. Sohrabi).

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Any language $\mathcal{L}$ consists of relation symbols, constant symbols and function symbols. The $\mathcal{L}$-structures can be interpreted in a language without function symbol, replacing each function with the relation associated to its graph. The notions of elementary equivalence for the two languages coincide.

For a pair of structures, for instance a ring R and an R-module M, it is convenient to use a language without function symbol and interpret each operation, for instance the action of R on M, as a relation on the disjoint union of the two structures. We obtain the same notion of elementary equivalence if we interpret each pair as a multi-based model, possibly with function symbols (see [3]).

According to the definition given by André Nies in [6], for each finite language $\mathcal{L}$ with function symbols, a finitely generated $\mathcal{L}$-structure is said to be *quasi-finitely axiomatizable* if it is characterized among finitely generated $\mathcal{L}$-structures by one sentence.

For each language $\mathcal{L}$ containing $+$ and each $\mathcal{L}$-structure M such that $(M,+)$ is an abelian group, we say that M is *finite dimensional over* $\mathbb{Z}$ or *FDZ* if $(M,+)$ is finitely generated. We say that M is *FDZ-finitely axiomatizable* if it is characterized among FDZ $\mathcal{L}$-structures by one sentence.

We note that, for each structure M and each finite sequence $\overline{x}$ which generates M (resp. $(M,+)$), the structure $(M,\overline{x})$ is quasi-finitely (resp. FDZ-finitely) axiomatizable if and only if there exists a formula $\varphi(\overline{u})$ satisfied by $\overline{x}$ in M such that, for each finitely generated (resp. FDZ) structure N and each $\overline{y}$ which satisfies φ in N, the map $\overline{x}\to\overline{y}$ induces an isomorphism from M to N.

Theorem 1.1. *Consider an FDZ scalar ring A and a finite sequence $\overline{a}$ which generates $(A,+)$. Then $(A,\overline{a})$ is FDZ-finitely axiomatizable.*

Theorem 1.2. *Consider an FDZ scalar ring A, an FDZ A-module M and some finite sequences $\overline{a}$ and $\overline{x}$ which generate $(A,+)$ and $(M,+)$. Then $(A,\overline{a},M,\overline{x})$ is FDZ-finitely axiomatizable.*

Theorem 1.3. *Consider an FDZ scalar ring A, an FDZ ring R equipped with a structure of A-module such that the multiplication of R is A-bilinear, and some finite sequences $\overline{a}$ and $\overline{x}$ which generate $(A,+)$ and $(R,+)$. Then $(A,\overline{a},R,\overline{x})$ is FDZ-finitely axiomatizable.*

For each ring $(R,+,.)$, we consider the two-sided ideals:

$$Ann(R)=\{x\in R\mid xy=yx=0 \text{ for each } y\in R\},$$

$$R^2=\{x_1y_1+\cdots+x_ny_n\mid n\in\mathbb{N} \text{ and } x_1,y_1,\ldots,x_n,y_n\in R\},$$

$$Is(I)=\{x\in R\mid nx\in I \text{ for some } n\in\mathbb{N}^*\} \text{ for each two-sided ideal } I \text{ of } R,$$

$$K(R)=Ann(R)+Is(R^2) \text{ and } L(R)=Is(Ann(R)+R^2).$$

We say that R is *regular* if $K(R)=L(R)$. Then, for each $S\subset R$ such that $Ann(R)=S\oplus(Ann(R)\cap Is(R^2))$, there exists $T\subset R$ containing $Is(R^2)$ such that $R=S\oplus T$.

We say that R is *tame* if $Ann(R)\subset Is(R^2)$.

The following result is a generalization of Theorem 1.1:

Theorem 1.4. *Consider a tame FDZ ring R and a finite sequence $\overline{x}$ which generates $(R,+)$. Then $(R,\overline{x})$ is FDZ-finitely axiomatizable.*

In contrast with the results above, we have:

Theorem 1.5. *For any FDZ rings R,S and each $k\in\mathbb{N}^*$ which is divisible by $|L(S)/K(S)|$, the following properties are equivalent:*

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