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What does a group algebra of a free group “know” about the group?

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ABSTRACT

We describe solutions to the problem of elementary classification in the class of group algebras of free groups. We will show that unlike free groups, two group algebras of free groups over infinite fields are elementarily equivalent if and only if the groups are isomorphic and the fields are equivalent in the weak second order logic. We will show that the set of all free bases of a free group F is 0-definable in the group algebra $K(F)$ when K is an infinite field, the set of geodesics is definable, and many geometric properties of F are definable in $K(F)$. Therefore $K(F)$ “knows” some very important information about F. We will show that similar results hold for group algebras of limit groups.

1. Introduction

This is the third paper (after [25], [24]) in a series of papers on the project on model theory of algebras outlined in our talk at the ICM in Seoul [23].

Tarski’s problems on groups, rings, and other algebraic structures were very inspirational and led to some important developments in modern algebra and model theory. Usually solutions to these problems for some structure clarify the most fundamental algebraic properties of the structure and give perspective on the expressive power of the first-order logic in the structure. We mention here results on first-order theories of algebraically closed fields, real closed fields [37], fields of p-adic numbers [1,14], abelian groups and modules

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[35,6], boolean algebras [36,13], free and hyperbolic groups [21,26,32,33], free associative algebras [25], group algebras [24].

In this paper we show that unlike free groups, two group algebras of free groups over infinite fields are elementarily equivalent if and only if the groups are isomorphic and the fields are equivalent in the weak second order logic (Theorem 16). We will prove that for a finitely generated free group F and infinite field K, if L is field and H is a group such that there is an element in H with finitely generated centralizer in $L(H)$, the group algebras $K(F)$ and $L(H)$ are elementarily equivalent ($K(F) \equiv L(H)$) if and only if H is isomorphic to F and the fields are equivalent in the weak second order logic (Theorem 14). Notice that for any field, the Diophantine problem, and, therefore, the theory of a group algebra of a torsion free hyperbolic group is undecidable [24]. Notice also that for any field K the first-order theory of the group algebra $K(F)$ is not stable because $K(F)$ is an integral domain (but not a field). We will prove the definability of bases of F in $K(F)$ when K is an infinite field (Theorem 9). In contrast to this, we proved that primitive elements, and, therefore, free bases are not definable in a free group of rank greater than two [22]. This was a solution of an old Malcev's problem, and the same question about group algebras is also attributed to Malcev. We will also show that the set of geodesics is definable, and many geometric properties of F are definable in $K(F)$ (Theorem 11). Hence $K(F)$ "knows" some very important information about F (this information can be expressed in the first order language) that F itself does not know. We will show that similar results hold for group algebras of limit groups (Theorems 11, 17, 18, 12). Recall, that a limit group is a finitely generated fully residually free group, namely a group G such that for any finite set of distinct elements in G there is a homomorphism into a free group that is injective on this set. We will also prove (Theorem 13) definability of finitely generated subrings, subgroups, ideals and submonoids in $K(G)$ for a non-abelian limit group G and infinite field K (in some cases K has to be interpretable in $\mathbb{N}$), and prove that the theory of $K(G)$ does not admit quantifier elimination to boolean combinations of formulas from Π_n or Σ_n (with constants from $K(G)$) for any bounded n (Theorem 19).

Our main tool in proving these is the method of first-order interpretation. We remind here some precise definitions and several known facts that may not be very familiar to algebraists.

A language L is a triple $(\mathcal{F}_L, \mathcal{P}_L, \mathcal{C}_L)$, where $\mathcal{F}_L = \{f, \ldots\}$ is a set of functional symbols f coming together with their arities $n_f \in \mathbb{N}$, $\mathcal{P}_L$ is a set of relation (or predicate) symbols $\mathcal{P}_L = \{P, \ldots\}$ coming together with their arities $n_P \in \mathbb{N}$, and a set of constant symbols $\mathcal{C}_L = \{c, \ldots\}$. Sometimes we write $f(x_1, \ldots, x_n)$ or $P(x_1, \ldots, x_n)$ to show that $n_f = n$ or $n_P = n$. Usually we denote variables by small letters $x, y, z, a, b, u, v, \ldots$, while the same symbols with bars $\bar{x}, \ldots$ denote tuples of the corresponding variables $\bar{x} = (x_1, \ldots, x_n), \ldots$. A structure in the language L (an L-structure) with the base set A is sometimes denoted by $\mathbb{A} = \langle A; L\rangle$ or simply by $\mathbb{A} = \langle A; f, \ldots, P, \ldots, c, \ldots\rangle$. For a given structure $\mathbb{A}$ by $L(\mathbb{A})$ we denote the language of $\mathbb{A}$. Throughout this paper we use frequently the following languages that we fix now: the language of groups $\{\cdot, ^{-1}, 1\}$, where $\cdot$ is the binary multiplication symbol, $^{-1}$ is the symbol of inversion, and 1 is the constant symbol for the identity; and the language of unitary rings $\{+, \cdot, 0, 1\}$ with the standard symbols for addition, multiplication, and the additive identity 0. When the language L is clear from the context, we follow the standard algebraic practice and denote the structure $\mathbb{A} = \langle A; L\rangle$ simply by A. For example, we refer to a field $\mathbb{F} = \langle F; +, \cdot, 0, 1\rangle$ simply as F, or to a group $\mathbb{G} = \langle G; \cdot, ^{-1}, 1\rangle$ as G, etc.

Let $\mathbb{B} = \langle B; L(\mathbb{B})\rangle$ be a structure. A subset $A \subseteq B^n$ is called *definable* in $\mathbb{B}$ if there is a formula $\phi(x_1, \ldots, x_n)$ in $L(\mathbb{B})$ such that $A = \{(b_1, \ldots, b_n) \in B^n \mid \mathbb{B} \models \phi(b_1, \ldots, b_n)\}$. In this case one says that ϕ defines A in $\mathbb{B}$. Similarly, an operation f on the subset A is definable in $\mathbb{B}$ if its graph is definable in $\mathbb{B}$. An n-ary predicate $P(x_1, \ldots, x_n)$ is definable in $\mathbb{B}$ if the set $\{(b_1, \ldots, b_n) \in \mathbb{B}^n | P(b_1, \ldots, b_n) \text{ is true}\}$ is definable in $\mathbb{B}$.

In the same vein an algebraic structure $\mathbb{A} = \langle A; f, \ldots, P, \ldots, c, \ldots\rangle$ is definable in $\mathbb{B}$ if there is a definable subset $A^* \subseteq B^n$ and operations $f^*, \ldots,$ predicates $P^*, \ldots,$ and constants $c^*, \ldots,$ on A^* all definable in $\mathbb{B}$ such that the structure $\mathbb{A}^* = \langle A^*; f^*, \ldots, P^*, \ldots, c^*, \ldots, \rangle$ is isomorphic to $\mathbb{A}$. For example, if Z is the center of a group G then it is definable as a group in G, the same for the center of a ring.

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