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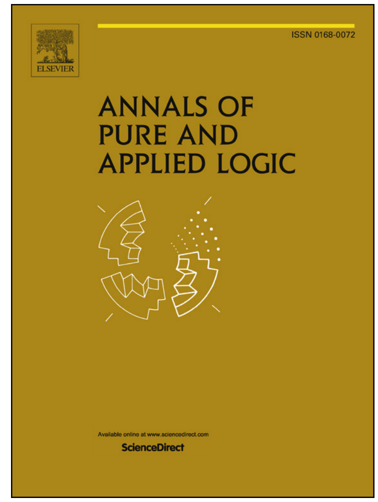

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TARSKI'S THEOREM ON INTUITIONISTIC LOGIC, FOR POLYHEDRA

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Abstract. In 1938, Tarski proved that a formula is not intuitionistically valid if, and only if, it has a counter-model in the Heyting algebra of open sets of some topological space. In fact, Tarski showed that any Euclidean space $\mathbb{R}^n$ with $n \geqslant 1$ suffices, as does e.g. the Cantor space. In particular, intuitionistic logic cannot detect topological dimension in the Heyting algebra of all open sets of a Euclidean space. By contrast, we consider the lattice of open subpolyhedra of a given compact polyhedron $P \subseteq \mathbb{R}^n$, prove that it is a locally finite Heyting subalgebra of the (non-locally-finite) algebra of all open sets of P, and show that intuitionistic logic is able to capture the topological dimension of P through the bounded-depth axiom schemata. Further, we show that intuitionistic logic is precisely the logic of formulæ valid in all Heyting algebras arising from polyhedra in this manner. Thus, our main theorem reconciles through polyhedral geometry two classical results: topological completeness in the style of Tarski, and Jaśkowski's theorem that intuitionistic logic enjoys the finite model property. Several questions of interest remain open. E.g., what is the intermediate logic of all closed triangulable manifolds?

1. Introduction

If X is any topological space, the collection $\mathscr{O}(X)$ of its open subsets is a (complete) Heyting algebra whose underlying order is given by set-theoretic inclusion. One can then interpret formulæ of intuitionistic logic into $\mathscr{O}(X)$ by assigning open sets to propositional atoms, and then extending the assignment to formulæ using the operations of the Heyting algebra $\mathscr{O}(X)$. A formula is true under such an interpretation just when it evaluates to X. In 1938, Tarski ([35], English translation in [36]) proved that intuitionistic logic is complete with respect to this semantics. Moreover, Tarski showed that one can considerably restrict the class C of spaces under consideration without impairing completeness. In particular, one can take $\mathsf{C} \coloneqq \{X \mid X \text{ is metrisable}\}$, and even $\mathsf{C} \coloneqq \{\mathbb{R}\}$ or $\mathsf{C} \coloneqq \{2^{\mathbb{N}}\}$, where $2^{\mathbb{N}}$ denotes the Cantor space. Tarski's result opened up a research area that continues to prosper to this day. Immediate descendants of [35] are the three seminal papers [24, 25, 26] by McKinsey and Tarski; [25, §3] offers a different proof of the main result of [35] in the dual language of closed sets and co-Heyting algebras. For an exposition of the different themes in spatial logic we refer to [2].

Intuitionistic logic has the finite model property. In 1936 Jaśkowski sketched a proof of this fact [19]; the first detailed exposition of the result[1] seems to be [31, Theorem 5.4] (see also [11, Theorem 2.57]). Algebraically, the finite model property

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[1] Though not exactly of the proof sketched by Jaśkowski: cf. [31, Lemma 5.3 and footnote (16)].