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Structure and enumeration theorems for hereditary properties in finite relational languages

C. Terry

Department of Mathematics, University of Maryland, 4176 Campus Drive, College Park, MD, 20742-4015, USA

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ABSTRACT

Given a finite relational language $\mathcal{L}$, a hereditary $\mathcal{L}$-property is a class of finite $\mathcal{L}$-structures which is closed under isomorphism and model theoretic substructure. This notion encompasses many objects of study in extremal combinatorics, including (but not limited to) hereditary properties of graphs, hypergraphs, and oriented graphs. In this paper, we generalize certain definitions, tools, and results form the study of hereditary properties in combinatorics to the setting of hereditary $\mathcal{L}$-properties, where $\mathcal{L}$ is any finite relational language with maximum arity at least two. In particular, the goal of this paper is to generalize how extremal results and stability theorems can be combined with well-known techniques and tools to yield approximate enumeration and structure theorems. We accomplish this by generalizing the notions of extremal graphs, asymptotic density, and graph stability theorems using structures in an auxiliary language associated to a hereditary $\mathcal{L}$-property. Given a hereditary $\mathcal{L}$-property $\mathcal{H}$, we prove an approximate asymptotic enumeration theorem for $\mathcal{H}$ in terms of its generalized asymptotic density. Further we prove an approximate structure theorem for $\mathcal{H}$, under the assumption of that $\mathcal{H}$ has a stability theorem. The tools we use include a new application of the hypergraph containers theorem (Balogh–Morris–Samotij [16], Saxton–Thomason [38]) to the setting of $\mathcal{L}$-structures, a general supersaturation theorem for hereditary $\mathcal{L}$-properties (also new), and a general graph removal lemma for $\mathcal{L}$-structures proved by Aroskar and Cummings in [5]. Similar results in the setting of multicolored graphs and hypergraphs were recently proved independently by Falgas-Ravry, O'Connel, Strömberg, and Uzzell [21].

1. Introduction

The study of hereditary properties of combinatorial structures is an important topic within the field of extremal combinatorics. Out of the many results in this line of research has emerged a pattern for how to prove approximate asymptotic enumeration and structure results. The aim of this paper is to provide a general framework in which to view these results and to formalize this pattern of proof.

E-mail address: cterry@umd.edu.

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1.1. Background

A nonempty class of graphs $\mathcal{P}$ is called a *hereditary graph property* if it is closed under isomorphism and induced subgraphs. Given a hereditary graph property $\mathcal{P}$, let $\mathcal{P}_n$ denote the set of elements of $\mathcal{P}$ with vertex set $[n]$. There has been extensive investigation into the properties of $\mathcal{P}_n$, where $\mathcal{P}$ is a hereditary property of graphs and n is large, see for instance [1,2,9–11,18,19,39]. The main questions addressed in these papers concern *enumeration* (finding an asymptotic formula for $|\mathcal{P}_n|$) and *structure* (understanding what properties elements of $\mathcal{P}_n$ have with high probability). Given a graph H, $\text{Forb}(H)$ (respectively $\text{Forb}_{ind}(H)$) is the class of finite graphs omitting H as a non-induced (respectively induced) subgraph. For any graph H, both $\text{Forb}(H)$ and $\text{Forb}_{ind}(H)$ are hereditary graph properties. Therefore, work on hereditary graph properties can be seen as generalizing the many structure and enumeration results about graph properties of the form $\text{Forb}(H)$ and $\text{Forb}_{ind}(H)$, for instance those appearing in [23,27,36,37,35]. From this perspective, the study of hereditary graph properties has been a central area of research in extremal combinatorics.

There are many results which extend the investigation of hereditary graph properties to other combinatorial structures. Examples of this include [14] for tournaments, [13] for oriented graphs and posets, [20,26] for k-uniform hypergraphs, and [24] for colored k-uniform hypergraphs. The results in [6,7,34,33] investigate asymptotic enumeration and structure results for specific classes of H-free hypergraphs, which are examples of hereditary properties of hypergraphs. Similarly, the results in [29] concern specific examples of hereditary properties of digraphs. The results in [32] for metric spaces are similar in flavor, although they have not been studied explicitly as instances of hereditary properties. Thus, extending the investigation of hereditary graph properties to other combinatorial structures has been an active area of research for many years.

From this investigation, patterns have emerged for how to prove these kinds of results, along with a set of standard tools, including extremal results, stability[1] theorems, regularity lemmas, supersaturation results, and the hypergraph containers theorem. In various combinations with extremal results, stability theorems, and supersaturation results, Szemerédi's regularity lemma has played a key role in proving many results in this area, especially those extending results for graphs to other settings. A sampling of these are [3,12,37] for graphs, [4] for oriented graphs, [6,7,20,24,26,33,34] for hypergraphs, and [32] for metric spaces. The hypergraph containers theorem, independently developed in [16,38], has been used in many recent papers in place of the regularity lemma. Examples of this include [17,15,25,31,16,38] for graphs, [29] for digraphs, and [8] for metric spaces. In these papers, the commonalities in the proofs are especially clear. Given an extremal result, there is clear outline for how to prove an approximate enumeration theorem. If on top of this, one can characterize the extremal structures and prove a corresponding stability theorem, then there is a clear outline for how to prove an approximate structure theorem. The goal of this paper is to make these proof outlines formal using generalizations of tools, definitions, and theorems from these papers to the setting of structures in finite relational languages.

1.2. Summary of results

Given a first-order language $\mathcal{L}$, we say a class $\mathcal{H}$ of $\mathcal{L}$-structures has the *hereditary property* if for all $A \in \mathcal{H}$, if B is a model theoretic substructure of A, then $B \in \mathcal{H}$.

Definition 1.1. Suppose $\mathcal{L}$ is a finite relational language. A *hereditary $\mathcal{L}$-property* is a nonempty class of finite $\mathcal{L}$-structures which has the hereditary property and which is closed under isomorphism.

This is the natural generalization of existing notions of hereditary properties of various combinatorial structures. Indeed, for appropriately chosen $\mathcal{L}$, almost all of the results cited so far are for hereditary

[1] This use of the word stability refers to a type of theorem from extremal combinatorics and is unrelated to the model theoretic notion of stability.