ARTICLE IN PRESS
APAL:2618

Annals of Pure and Applied Logic ••• (••••) •••–•••

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Recognizable sets and Woodin cardinals: computation beyond the constructible universe ☆

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ARTICLE INFO

Article history:
Received 19 December 2015
Received in revised form 10 October 2017
Accepted 6 December 2017
Available online xxxx

MSC:
03D10
03D32
03E15
03E45

Keywords:
Infinite time Turing machines
Algorithmic randomness
Effective descriptive set theory
Woodin cardinals
Inner models

ABSTRACT

We call a subset of an ordinal λ *recognizable* if it is the unique subset x of λ for which some Turing machine with ordinal time and tape and an ordinal parameter, that halts for all subsets of λ as input, halts with the final state 0. Equivalently, such a set is the unique subset x which satisfies a given Σ_1 formula in $L[x]$. We further define the *recognizable closure* for subsets of λ by closing under relative recognizability for subsets of λ.
We prove several results about recognizable sets and their variants for other types of machines. Notably, we show the following results from large cardinals.

- Recognizable sets of ordinals appear in the hierarchy of inner models at least up to the level Woodin cardinals, while computable sets are elements of L.
- A subset of a countable ordinal λ is in the recognizable closure for subsets of λ if and only if it is an element of the inner model M^∞, which is obtained by iterating the least measure of the least fine structural inner model M_1 with a Woodin cardinal through the ordinals.

1. Introduction

Infinitary machine models of computation provide an attractive approach to generalized recursion theory. The first such model, Infinite Time Turing Machines, was introduced by Hamkins and Lewis [15]. A moti-

☆ Work on this paper was partially done whilst the authors were visiting at the Isaac Newton Institute for Mathematical Sciences in the programme 'Mathematical, Foundational and Computational Aspects of the Higher Infinite' (HIF), to which they are grateful. This program was funded by EPSRC grant EP/K032208/1. In addition the second author was partially supported by DFG-grant LU2020/1-1. Moreover, the last author was a Simons Foundation Fellow during this period and gratefully acknowledges that Foundation's support.
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https://doi.org/10.1016/j.apal.2017.12.007

vation for considering such machine models is that they capture the notion of an effective procedure in a more general sense than classical Turing machines, thus allowing effective mathematics of the uncountable (see [13] for other approaches on this topic). Such models are usually obtained by extending the working time or the working space of a classical model of computation to the transfinite. The strongest such models considered so far, to our knowledge, are Ordinal Turing Machines (OTMs) and the equivalent Ordinal Register Machines (ORMs). These were defined and studied by Peter Koepke and others [21,23]. It is argued in [2] that OTM-computability adequately expresses the intuitive notion of an idealized computor working in transfinite time and space.

The sets of ordinals which are OTM-computable from ordinal parameters are simply the constructible sets of ordinals. This is rather restrictive and it was asked whether one should study machines that have an extra function allowing them to go outside of L into core models [9]. This idea suggests a strengthening of the underlying machine model.

Here we follow a different approach and consider the notion of recognizability. This means that for some initial input, some program will stop with output 1 if the input is the object in question, and stop with output 0 otherwise. It is thus a form of implicit definability. The fact that recognizability is strictly weaker than computability was first noticed for Infinite Time Turing Machines and is called the *lost melody phenomenon* [15] (see also [1]). This roughly means that implicit computability (and implicit definability) are different from computability (and definability). Moreover, it should be stressed that recognizability is equivalent to Σ_1-definability in models of the form $\mathrm{L}[x]$, where x is the input of the computation (see Lemma 2.3).

The notion of recognizability was independently first considered for OTM-computability by Dawson [6]. He showed that the OTM-computable sets coincide with the recognizable sets, without allowing ordinal parameters. Moreover, he showed that if this was relaxed to allow constructibly countable ordinal parameters, then every recognizable set is still constructible. He further showed that $0^\sharp$ (see e.g. [18, Section 9], [29, Definition 10.37]), if it exists, is recognizable from some uncountable cardinal, and that adding Cohen reals over L does not add recognizable sets. In fact, an OTM can recognize $0^\sharp$ from the parameter ω_1 [1], although $0^\sharp$ is not constructible.

As for computability, it is natural to study relativized recognizability. Intuitively, an object x is recognizable relative to an object y if this can be used to identify x. This concept is illustrated by our inability to recognize a radioactive stone, while it is possible to recognize a Geiger counter and use this to identify the stone. It is moreover natural to iterate relative recognizability in finitely many steps and thus obtain the *recognizable closure* $\mathcal{C}$ (see Definition 2.6).

We attempt a systematic study of recognizability, its variants, and their relationships to other kinds of implicit definability. Moreover, we study the recognizable closure and particularly its relationship to HOD and other well-known inner models. Thus we address the following questions.

- How does recognizability change with different ordinal parameters?
- How can the recognizable closure be characterized?

The results of this paper show that while recognizability from fixed ordinal parameters is not absolute to generic extensions, the recognizable closure is more stable, assuming the existence of large cardinals.

We now describe some of the results. Since the recognizable sets have not been studied before, we include various fundamental facts, many of which are easy to prove.

It is surprising that there is a close connection between recognizability and the notion of *implicit definability* that was introduced by Hamkins and Leahy in [14]. They define a set x of an ordinal α to be *implicitly definable over* L if there is a formula $\varphi(y, \beta)$ and an ordinal γ such that x is the unique subset z of α with $\langle \mathrm{L}, \in, z\rangle \vDash \varphi(z, \gamma)$. We obtain the following equivalence in Theorem 3.12.

Please cite this article in press as: M. Carl et al., Recognizable sets and Woodin cardinals: computation beyond the constructible universe, Ann. Pure Appl. Logic (2017), https://doi.org/10.1016/j.apal.2017.12.007