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An analysis of the logic of Riesz spaces with strong unit

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A B S T R A C T

We study Łukasiewicz logic enriched by a scalar multiplication with scalars in $[0,1]$. Its algebraic models, called *Riesz MV-algebras*, are, up to isomorphism, unit intervals of Riesz spaces with strong unit endowed with an appropriate structure. When only rational scalars are considered, one gets the class of *DMV-algebras* and a corresponding logical system. Our research follows two objectives. The first one is to deepen the connections between functional analysis and the logic of Riesz MV-algebras. The second one is to study the finitely presented MV-algebras, DMV-algebras and Riesz MV-algebras, connecting them from logical, algebraic and geometric perspective.

Introduction

In this paper we present the logical system $\mathcal{R}$Ł which extends the infinitely valued Łukasiewicz logic with a family of unary operators that are semantically interpreted as scalar multiplication with scalars from the real interval $[0,1]$. The category of the corresponding algebraic structures is equivalent with the category of Riesz spaces with strong unit.

Recall that Łukasiewicz logic Ł is the system that has $\{\rightarrow, \neg\}$ as basic connectives and whose axioms are L1–L4 below:

(L1) $\varphi \rightarrow (\psi \rightarrow \varphi)$

(L2) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

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(L3) $(\varphi \vee \psi) \to (\psi \vee \varphi)$
(L4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$.
The only deduction rule is *modus ponens*.

The corresponding algebraic structures, MV-algebras, were defined by C.C. Chang in 1958 [7]. Chang's definition was inspired by the theory of lattice-ordered groups, consequently MV-algebras are structures $(A, \oplus, \neg, 0)$ satisfying some appropriate axioms, where $x \to y = \neg x \oplus y$ for any x, y. The connection between MV-algebras and Abelian lattice-ordered groups was fully investigated by D. Mundici [28] who proved the fundamental result that MV-algebras are categorically equivalent with Abelian lattice-ordered groups with strong unit.

Since the standard model of Ł is the real interval $[0, 1]$ endowed with the Łukasiewicz negation $\neg x = 1 - x$ and the Łukasiewicz implication $x \to y = \min(1 - x + y, 1)$, a natural problem was to study Łukasiewicz logic enriched with a product operation, semantically interpreted in the real product on $[0, 1]$. This line of research led to the definition of PMV-algebras, which are MV-algebras endowed with an internal binary operation, but in this case the standard model only generates a proper subvariety. Through an adaptation of Mundici's equivalences, PMV-algebras and their logic are connected with the theory of lattice-ordered rings with strong unit.

A different approach is presented in [12,16], where the real product on $[0, 1]$ is interpreted as a scalar multiplication with scalars taken in $[0, 1]$. The system $\mathcal{R}$Ł further developed in this paper is a relatively simple extension of Ł which is obtained by adding to the infinitely valued Łukasiewicz logic a family of unary operators $\{\nabla_r\}_{r\in[0,1]}$, whose dual operators are semantically interpreted in a scalar multiplication. Consequently, Riesz MV-algebras – the corresponding algebraic structures – are categorically equivalent with Riesz spaces with a strong unit. Note that our results are not the first connection between Łukasiewicz logic and the theory of Riesz spaces, one can see [2,18] for previous investigations, while the seminal idea of a connection between Riesz spaces and a subcategory of MV-algebras was given in [14].

In this paper, we establish connections between the system $\mathcal{R}$Ł and elements of functional analysis, where Riesz spaces are fundamental structures. After some needed preliminaries, in Section 2 we define $\mathcal{R}$Ł and prove some logic-related results, the main being a syntactical characterization of uniform convergence in a Riesz space. Using this concept of *limit of formulas*, in Theorem 2.12 we describe any formula in $\mathcal{R}$Ł as a sequence of formulas in the Rational Łukasiewicz logic $\mathcal{Q}$Ł [17] and in Section 2.4 we characterize two norm-completions of the Lindenbaum–Tarski algebra of $\mathcal{R}$Ł.

In Section 3 we establish categorical equivalences between subcategories of finitely presented MV-algebras, DMV-algebras and Riesz MV-algebras. In doing so we propose three different approaches, that take advantage of state-of-the-art techniques on MV-algebras: polyhedra, tensor product and categories of presentations. Finally, we link all results together in Section 3.4, where we use the syntactical notion of limit to fully describe those theories of $\mathcal{R}$Ł that are axiomatized by formulas of Ł.

1. Preliminaries on algebraic structures

MV-algebras, the algebraic counterpart of the ∞-valued Łukasiewicz propositional logic Ł, are structures $(A, \oplus, {}^*, 0)$ of type $(2, 1, 0)$ such that $(A, \oplus, 0)$ is an Abelian monoid and the following identities hold for any $x, y \in A$:

$$(x^*)^* = x, \quad (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x, \quad 0^* \oplus x = 0^*.$$

Further operations are defined as follows: 1 is 0^*, Łukasiewicz implication is $x \to y = x^* \oplus y$, Łukasiewicz conjunction is $x \odot y = (x^* \oplus y^*)^*$ and Chang's distance is $d(x, y) = (x^* \odot y) \oplus (x \odot y^*)$, for any x, $y \in A$. If $x \vee y = x \oplus (y \odot x^*)$ and $x \wedge y = (x^* \vee y^*)^*$ then $(A, \vee, \wedge, 1, 0)$ is a bounded distributive lattice. If $x \in A$ and $n \in \mathbb{N}$ then $0x = 0$ and $(n + 1)x = (nx) \oplus x$.