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Natural deduction for bi-intuitionistic logic

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A B S T R A C T

We present a multiple-assumption multiple-conclusion system for bi-intuitionistic logic. Derivations in the systems are graphs whose edges are labelled by formulas and whose nodes are labelled by rules. We show how to embed both the standard intuitionistic and dual-intuitionistic natural deduction systems into the proposed system. Soundness and completeness are established using translations with more traditional sequent calculi for bi-intuitionistic logic.

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1. Introduction

Bi-intuitionistic logic (also known as subtractive or H-B logic) is a conservative extension of intuitionistic logic obtained by the addition of a new connective: co-implication $\notin$.

Whereas implication relates to conjunction as follows:

$$A \wedge B \vdash C \quad \text{iff} \quad A \vdash B \supset C$$

co-implication relates to disjunction as follows[2]:

$$A \vdash B \vee C \quad \text{iff} \quad A \notin B \vdash C$$

In the $\{\wedge, \vee, {\sim}\}$-fragment of classical logic both implication and co-implication can be defined (respectively as ${\sim}A \vee B$ and $A \wedge {\sim}B$). On the other hand, in intuitionistic logic implication is independent from the

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[2] We thereby stick to the terminology of Goré [9], Buisman and Goré [2], Goré et al. [10], Wansing [31], Pinto and Uustalu [14,15] (who indicate co-implication using the sign $\prec$), rather than of Wolter [34], Schroeder-Heister [23] and Tranchini [26], (who use the same sign $\prec$ to refer to the converse of our co-implication). Co-implication (as understood in the present paper) is sometimes referred to as pseudo-difference $A \dot{-} B$, see for instance [18,19,8,30], sometimes as subtraction, noted as $A - B$ in [21,3,4] and as $A \backslash B$ in [1]. The adoption of the so-far unused sign $\notin$ is meant as a help towards the possible confusion.

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connectives $\{\wedge, \vee, \neg\}$ [12]. A well-known result (see, e.g., [30], Theorem 5.2) is that also co-implication is undefinable in terms of the intuitionistic connectives $\{\supset, \wedge, \vee, \bot\}$.

As implication is the distinctive connective of intuitionistic logic, the natural habitat of co-implication is dual-intuitionistic logic.

A sequent calculus for intuitionistic logic `LI` can be obtained by restricting all sequents in the calculus for classical logic `LK` to at most one formula in the succedent. A sequent calculus for dual-intuitionistic logic `LDI` can be obtained by imposing the dual restriction to the sequents of `LK`: at most one formula in the antecedent.

Let Γ and Δ be multi-sets of formulas, with $|\Delta| \leq 1$. The correspondence between the natural deduction system for intuitionistic logic `NI` and `LI` can be roughly stated as follows: a sequent $\Gamma \Rightarrow \Delta$ is derivable in `LI` if and only if (iff) there is a derivation in `NI` whose undischarged top-formulas are in Γ and whose bottom-formula is the only element of Δ (if $|\Delta| = 1$, $\bot$ otherwise).

In [26] a multiple-conclusion and single-premise natural deduction system for dual-intuitionistic logic `NDI` was defined by turning `NI` upside-down and exchanging each connective with its dual ($\supset$ with $\not\subset$, $\wedge$ with $\vee$ and viceversa, $\bot$ with $\top$). As a result, a sequent $\Delta \Rightarrow \Gamma$ is derivable in `LDI` iff there is a derivation in `NDI` whose undischarged bottom-formulas are in Γ and whose top-formula is the only element of Δ (if $|\Delta| = 1$, $\top$ otherwise).

Contrary to dual-intuitionistic logic, the proof-theory of bi-intuitionistic logic proved quite tricky to characterize, due to the presence of both implication and co-implication. As of today, all "plain" sequent calculi for bi-intuitionistic logic presented in the literature [18,21,3] has been shown not to enjoy cut-elimination (although several cut-free "enriched" sequent calculi, using nested [10] or labelled sequents [14,15], as well as display calculi [9,31] have been recently developed).

Concerning natural deduction, although a few authors hinted at how to extend the natural deduction system for intuitionistic logic with rules for co-implication [13,11], the only fully fledged natural deduction approach to bi-intuitionistic logic is that of [4]. Crolard's system, however, is not in Prawitz's style natural deduction, but in the sequent-style format. Sequents have a multiple-antecedents/multiple-succedent format, but a relation between formulas in the antecedents and in the consequents is essential to properly formulate certain restrictions on the rules.

In the present paper, we propose a multiple-premise multiple-conclusion natural deduction for bi-intuitionistic logic. Its key feature is its being fully bi-directional, that is its derivations can be built as much starting from the top (as in `NI`) or from the bottom (as in `NDI`). The idea of multiple-premise multiple-conclusion systems of natural deduction goes back at least to [24,29] and thanks to the work in linear logic initiated by Girard [7] has become a leading paradigm for proof systems. Rather than linear logic proof nets, the system here presented is closer to the N-graphs introduced by de Oliveira in her PhD thesis [5] and subsequently investigated in joint work with de Queiroz and others (see, e.g., [17]) (we became aware of this line of work in the process of revising the present paper), and to the "circuit proofs" of Restall [22] (whose graphic style we will quite closely follow).

2. Philosophical remarks on proofs and refutations

2.1. Proofs in intuitionistic logic

Following ideas first put forward by Prawitz and Martin-Löf (see, e.g., [16], §I.3.5.6), we view derivations as formal objects representing a particular kind of abstract entities called *constructions*. Constructions are here understood in accordance with the intuitionistic philosophy of mathematics, as the result of the performance of certain operations by an idealized knowing subject.

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