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The proof-theoretic strength of Ramsey's theorem for pairs and two colors

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ABSTRACT

Ramsey's theorem for n-tuples and k-colors (RT^n_k) asserts that every k-coloring of $[\mathbb{N}]^n$ admits an infinite monochromatic subset. We study the proof-theoretic strength of Ramsey's theorem for pairs and two colors, namely, the set of its Π^0_1 consequences, and show that RT^2_2 is Π^0_3 conservative over $\mathrm{I}\Sigma^0_1$. This strengthens the proof of Chong, Slaman and Yang that RT^2_2 does not imply $\mathrm{I}\Sigma^0_2$, and shows that RT^2_2 is finitistically reducible, in the sense of Simpson's partial realization of Hilbert's Program. Moreover, we develop general tools to simplify the proofs of Π^0_3-conservation theorems.

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1. Introduction

Ramsey's theorem for n-tuples and k-colors (RT^n_k) asserts that every k-coloring of $[\mathbb{N}]^n$ admits an infinite monochromatic subset. Ramsey's theorem is probably the most famous theorem of Ramsey's theory, and plays a central role in combinatorics and graph theory (see, e.g., [29,26]) with numerous applications in mathematics and computer science, among which functional analysis [2] automata theory [54], or termination analysis [55]. An important aspect of Ramsey's theorem is its definable class of fast-growing functions. Erdös [21] showed that the (diagonal) Ramsey number has an exponential growth rate. Actually, Ramsey's theorem defines much faster-growing functions, which is studied by Ketonen and Solovay [38], among others. The growth rate of these functions have important applications, since it provides upper bounds to combinatorial questions from various fields. This type of question is heavily related to proof theory, and with their language, the question is formalized as follows:

> *What is the class of functions whose existence is provable (with an appropriate base system) from Ramsey's theorem?*

For example, the Ramsey number function belongs to this class since the existence of the Ramsey number $R(n, k)$ is guaranteed by Ramsey's theorem. In fact, this class of functions decides the so-called "proof-theoretic strength" of Ramsey's theorem.

Ramsey's theorem also plays a very important role in reverse mathematics as it is one of the main examples of theorems escaping the Big Five phenomenon (see Section 1.2). Reverse mathematics is a general program that classifies theorems by two different measures, namely, by their computability-theoretic strength and by their proof-theoretic strength. As it happens, consequences of Ramsey's theorem are notoriously hard to study in reverse mathematics, and therefore received a lot of attention from the reverse mathematics community. Especially, determining the strength of Ramsey's theorem for pairs (RT^2_2) is always a central topic in the study of reverse mathematics. This study yielded series of seminal papers [35,56,13,14] introducing both new computability-theoretic and proof-theoretic techniques. (See Section 1.2 for more details of its computability-theoretic strength.)

In this paper, we mainly focus on the proof-theoretic strength of Ramsey's theorem for pairs. By the proof-theoretic strength of a theory T we mean the set of Π^0_1 sentences which are provable in T, or the proof-theoretic ordinal of T which is decided by the class of (Σ^0_1-definable) functions whose totality are proved in T. In fact, we will give the exact proof-theoretic strength of RT^2_2 by proving that $\mathrm{RT}^2_2 + \mathsf{WKL_0}$ is a Π^0_3-conservative extension of $\mathrm{I}\Sigma^0_1$ (Theorem 7.4), where $\mathsf{WKL_0}$ stands for weak König's lemma and $\mathrm{I}\Sigma^0_n$ is the Σ^0_n-induction scheme. This answers the long-standing open question of determining the Π^0_2-consequences of RT^2_2 or the consistency strength of RT^2_2, posed, e.g., in Seetapun and Slaman [56, Question 4.4] Cholak, Jockusch and Slaman [13, Question 13.2] Chong and Yang [17] (see Corollaries 7.5 and 7.6). For this, we use a hybrid of forcing