Lie algebras/Differential geometry

Geodesic orbit metrics on compact simple Lie groups arising from flag manifolds

Métriques définies par les variétés de drapeaux sur les groupes de Lie compacts, simples, dont les géodésiques sont des orbites

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ABSTRACT

In this paper, we investigate left-invariant geodesic orbit metrics on connected simple Lie groups, where the metrics are formed by the structures of flag manifolds. We prove that all these left-invariant geodesic orbit metrics on simple Lie groups are naturally reductive.

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RÉSUMÉ

Dans cet article, nous étudions les métriques à géodésiques homogènes, invariantes à gauche, sur des groupes de Lie simples connexes, où les métriques sont définies par les structures de variétés de drapeaux. Nous montrons que toutes ces métriques à géodésiques homogènes invariantes à gauche sur des groupes de Lie simples sont naturellement réductives.

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1. Introduction

Consider a homogeneous Riemannian manifold $(M = G/H, g)$, where H is a compact subgroup of G and g is a G-invariant Riemannian metric on M. If every geodesic of M is the orbit of some 1-parameter subgroup of G, then M is called a *geodesic orbit space* (g.o. space), and the metric g is called a *geodesic orbit metric* (g.o. metric). A complete Riemannian manifold (M, g) is called *geodesic orbit* if it is a geodesic orbit space with respect to the isometry group. This terminology was introduced by O. Kowalski and L. Vanhecke in [9], where they started a systematic research program on geodesic orbit manifolds including the classification in dimensions ≤ 6.

After that, classifications were worked out under various settings. See [10], [13], [6] and their references.

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In [11], Nikonorov started to investigate g.o. metrics on compact simple Lie groups G with isometry group $G \times K$, where K is a compact subgroup of G. He obtained an equivalent algebraic condition for g.o. spaces. In [7], it was shown that all the g.o. metrics on compact Lie groups, arising from generalized Wallach spaces, are naturally reductive.

In this paper, we investigate all the geodesic orbit metrics on compact simple Lie groups G with the structure from flag manifolds. Using the structure of flag manifolds, we prove that all such g.o. metrics are naturally reductive with respect to $G \times K$.

This paper is organized as follows. In Section 2, we recall the definition and structure of flag manifolds, along with some basic facts on g.o. metrics on compact simple Lie groups. In Section 3, we prove that all these g.o. metrics are naturally reductive by using the structure of flag manifolds.

2. Geodesic orbit metrics on compact simple Lie groups and flag manifolds

In this paper, the Lie groups G and K are always assumed to be connected.

We first recall some basic concepts. Let K be a closed subgroup of Lie group G, a G-invariant metric g on $M = G/K$ corresponds to an $Ad(K)$-invariant scalar product $(\ ,\ )$ on $\mathfrak{m} = T_o M$ and vice versa. The metric g is called *standard* if the scalar product $(\ ,\ )$ on $\mathfrak{m}$ is the restriction of B, where B is the negative of the Killing form of $\mathfrak{g}$. For a given non-degenerate $Ad(K)$-invariant scalar product $(\ ,\ )$ on $\mathfrak{m}$, there exists an $Ad(K)$-invariant positive definite symmetric operator A on $\mathfrak{m}$ such that $(x, y) = B(Ax, y)$ for $x, y \in \mathfrak{m}$. Conversely, any such operator A determines an $Ad(K)$-invariant scalar product $(x, y) = B(Ax, y)$ on $\mathfrak{m}$. We call such A a *metric endomorphism*. A homogeneous Riemannian metric on $M = G/K$ is called *naturally reductive* if

$$([Z, X]_{\mathfrak{m}}, Y) + (X, [Z, Y]_{\mathfrak{m}}) = 0, \forall X, Y, Z \in \mathfrak{m}.$$

In [2], there is an equivalent algebraic description of g.o. metrics on $M = G/K$, which we recall below.

Theorem 2.1 ([2] *Corollary 2*). *Let* $(M = G/K, g)$ *be a homogeneous Riemannian manifold. Then M is geodesic orbit space if and only if, for every $X \in \mathfrak{m}$, there exists an $a(X) \in \mathfrak{k}$ such that*

$$[a(X) + X, AX] \in \mathfrak{k},$$

where A is the metric endomorphism.

According to the Ochiai–Takahashi theorem [12], the full connected isometry group $\mathrm{Isom}(G, g)$ of a simple compact Lie group G with a left-invariant Riemannian metric g is contained in the group $L(G)R(G)$, the product of left and right translations. Hence G is a normal subgroup in $\mathrm{Isom}(G, g)$, which is locally isomorphic to the group $G \times K$, where K is a closed subgroup of G, with action $(a, b)(c) = acb^{-1}$, where $a, c \in G$ and $b \in K$.

In [3], Alekseevski and Nikonorov showed that, if we choose G as the isometry group of the compact Lie group G with a left-invariant Riemannian metric, then we have the following Proposition.

Proposition 2.2 ([3] *Proposition 8*). *A compact Lie group G with a left-invariant metric g is a g.o. space if and only if the corresponding Euclidean metric $(\ ,\ )$ on the Lie algebra $\mathfrak{g}$ is bi-invariant.*

In [11], Nikonorov consider the isometry group of a compact simple Lie group G as $G \times K$, where K is a closed subgroup of G. Then he obtained the equivalent algebraic description of g.o. metrics g on compact simple Lie groups G as follows.

Theorem 2.3 ([11] *Proposition 10*). *Let (G, g) be a compact simple Lie group with a left-invariant Riemannian metric. Then the following are equivalent:* (i) *(G, g) is a geodesic orbit manifold,* (ii) *there is a closed connected subgroup K of G such that, for any $X \in \mathfrak{g}$, there is $W \in \mathfrak{k}$ such that $([X + W, Y], X) = 0$ for every $Y \in \mathfrak{g}$ and* (iii) *$[A(X), X + W] = 0$, where $A : \mathfrak{g} \to \mathfrak{g}$ is the metric endomorphism for (G, g).*

Let B denote the negative of the Killing form of $\mathfrak{g}$, the Lie algebra of G. Then we have an inner product on $\mathfrak{g}$ given by

$$(\ ,\ ) = A_0 B(\ ,\ )|_{\mathfrak{k}_0} + x_1 B(\ ,\ )|_{\mathfrak{k}_1} + \cdots + x_p B(\ ,\ )|_{\mathfrak{k}_p} + y_1 B(\ ,\ )|_{\mathfrak{m}_1} + \cdots + y_q B(\ ,\ )|_{\mathfrak{m}_q}, \tag{2.1}$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \mathfrak{k}_p$ is the decomposition of $\mathfrak{k}$ into non-isomorphic simple ideals and center, $\mathfrak{m}$ is the B-orthogonal complement of $\mathfrak{k}$ and $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ is the decomposition of $\mathfrak{m}$ into irreducible and mutually inequivalent $Ad(K)$-modules.

D'Atri and Ziller [8] have investigated naturally reductive metrics among the left-invariant metrics on compact Lie groups, and have given a complete classification in the case of simple Lie groups. The following is a description of naturally reductive left-invariant metrics on a compact simple Lie group (Theorem 2.4).