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On the shape factor of interaction laws for a non-local approximation of the Sobolev norm and the total variation

Sur le facteur de forme des lois d'interaction pour une approximation non locale de la norme de Sobolev et de la variation totale

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A R T I C L E   I N F O

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A B S T R A C T

We consider the family of non-local and non-convex functionals introduced by H. Brézis and H.-M. Nguyen in a recent paper. These functionals Gamma-converge to a multiple of the Sobolev norm or the total variation, depending on a summability exponent, but the exact values of the constants are unknown in many cases.
We describe a new approach to the Gamma-convergence result that leads in some special cases to the exact value of the constants, and to the existence of smooth recovery families.

R É S U M É

Nous considérons la famille des fonctionnelles non locales et non convexes introduites par H. Brézis et H.-M. Nguyen dans un article récent. Ces fonctionelles Gamma-convergent vers un multiple de la norme de Sobolev ou de la variation totale, en fonction d'un exposant de sommabilité, mais les valeurs exactes des constantes sont inconnues dans de nombreux cas.
Nous décrivons une nouvelle approche pour le résultat de Gamma-convergence, qui conduit, dans certains cas particuliers, à la valeur exacte des constantes et à l'existence de familles optimales régulières.

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1. Introduction

In a recent paper [6], H. Brézis and H.-M. Nguyen introduced the family of non-local functionals

$$\Lambda_{\delta,p}(\varphi,u,\Omega) := \iint_{\Omega^2} \varphi\left(\frac{|u(y)-u(x)|}{\delta}\right)\frac{\delta^p}{|y-x|^{d+p}}\,dx\,dy, \tag{1}$$

where d is a positive integer, $\Omega \subseteq \mathbb{R}^d$ is an open set, $\delta > 0$ is a real parameter, $p \geq 1$ is a real number, $u : \Omega \to \mathbb{R}$ is a measurable function, and $\varphi : [0,+\infty) \to [0,+\infty)$ is a measurable function that describes the extent to which a pair $(x,y) \in \Omega^2$ contributes to the double integral (1). For this reason, in the sequel, we call φ the "interaction law". A general enough class of admissible interaction laws is the set $\mathcal{A}$ of all functions $\varphi : [0,+\infty) \to [0,+\infty)$ that are not identically equal to zero and such that

- φ is nondecreasing and lower semicontinuous on $[0,+\infty)$, and actually continuous except at a finite number of points in $(0,+\infty)$,
- there exists a constant a such that $\varphi(t) \leq at^2$ for every $t \in [0,1]$,
- there exists a constant b such that $\varphi(t) \leq b$ for every $t \geq 0$.

The basic example is when $\varphi(x)$ coincides with

$$\varphi_k(x) := \begin{cases} 0 & \text{if } x \in [0,k], \\ 1 & \text{if } x > k, \end{cases} \tag{2}$$

where k is a positive real number.

The asymptotic behavior of the family $\Lambda_{\delta,p}$, starting with the model case where $\varphi = \varphi_1$, was investigated in a series of papers [3–6,8–11]. The general idea is that $\Lambda_{\delta,p}(\varphi,u,\Omega)$ is proportional, in the limit as $\delta \to 0^+$, to the functional

$$\Lambda_{0,p}(u,\Omega) := \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^p\,dx & \text{if } p > 1 \text{ and } u \in W^{1,p}(\Omega), \\ \text{total variation of } u \text{ in } \Omega & \text{if } p = 1 \text{ and } u \in BV(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

The first result in this direction concerns the pointwise limit, at least in the case of smooth functions with compact support. Indeed, for every admissible interaction law $\varphi \in \mathcal{A}$ it turns out that

$$\lim_{\delta\to 0^+} \Lambda_{\delta,p}\left(\varphi,u,\mathbb{R}^d\right) = G_{d,p}\cdot N(\varphi)\cdot \Lambda_{0,p}\left(u,\mathbb{R}^d\right) \qquad \forall u \in C^1_c(\mathbb{R}^d), \tag{4}$$

where $G_{d,p}$ is a geometric constant, and $N(\varphi)$ is a normalization constant, which we call the "scale factor" of the interaction law φ. These two constants are defined as (v is any element of the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$)

$$G_{d,p} := \frac{1}{p}\int_{\mathbb{S}^{d-1}} |\langle v,\sigma\rangle|^p\,d\sigma, \qquad N(\varphi) := \int_0^{+\infty} \frac{\varphi(t)}{t^2}\,dt. \tag{5}$$

The equality in (4) holds true also for every $u \in W^{1,p}(\mathbb{R}^d)$ if $p > 1$ (but not necessarily if $p = 1$).

The surprise comes with the Gamma-limit, which is expected to be of the form

$$\Gamma\text{–}\lim_{\delta\to 0^+} \Lambda_{\delta,p}\left(\varphi,u,\mathbb{R}^d\right) = G_{d,p}\cdot N(\varphi)\cdot K_{d,p}(\varphi)\cdot \Lambda_{0,p}\left(u,\mathbb{R}^d\right) \qquad \forall u \in L^p(\mathbb{R}^d), \tag{6}$$

where $K_{d,p}(\varphi) \in (0,1]$ is a suitable constant, whose appearance was defined in [6] as "mysterious and somewhat counterintuitive". This result was proved in [11] in the special case $\varphi = \varphi_1$ with general exponent $p \geq 1$, and in [6] for general $\varphi \in \mathcal{A}$ but a special exponent $p = 1$. As far as we know, the case with general interaction law $\varphi \in A$ and general exponent $p \geq 1$ has never been written explicitly, even if a paper in this direction was announced in [6].

The constant $K_{d,p}(\varphi)$ is invariant by both horizontal and vertical rescaling; namely it does not change when we replace $\varphi(t)$ with $\alpha\varphi(\beta t)$ for some positive constants α and β. For this reason, we call it the "shape factor" of the interaction law φ.

Computing shape factors is a difficult task, even in dimension one (and actually we think that they never depend on d). In this note, we describe a new approach to the Gamma-convergence problem that was carried out in our recent papers [1,2], and allowed us to compute the shape factor of some special interaction laws. In the sequel