Doctopic: Combinatorics ARTICLE IN PRESS CRASS1:6120

C. R. Acad. Sci. Paris, Ser. I ••• (••••) •••–•••

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Combinatorics

Symmetries on plabic graphs and associated polytopes

Symétries dans les graphes plan bicolores et les polytopes associés

Xin Fang [a], Ghislain Fourier [b]

[a] *University of Cologne, Mathematical Institute, Weyertal 86–90, 50931 Cologne, Germany*

[b] *Leibniz Universität Hannover, Institute for Algebra, Number Theory and Discrete Mathematics, Welfengarten 1, 30167 Hannover, Germany*

ARTICLE INFO

Article history:
Received 19 April 2018
Accepted 14 May 2018
Available online xxxx

Presented by Michele Vergne

ABSTRACT

For Grassmann varieties, we explain how the duality between the Gelfand–Tsetlin polytopes and the Feigin–Fourier–Littelmann–Vinberg polytopes arises from different positive structures.

RÉSUMÉ

Nous expliquons, pour les variétés grasmanniennes, comment la dualité entre les polytopes de Gelfand–Tsetlin et les polytopes de Feigin–Fourier–Littelman–Vinberg émerge dans différentes structures positives.

1. Introduction

Plabic graphs (planar bicoloured graphs) were introduced by Postnikov [8] to parametrize cells in the totally non-negative (TNN) Grassmannians $(\mathrm{Gr}_{k,n}(\mathbb{R}))_{\geq 0}$. These graphs are drawn inside a disk with boundary vertices labelled by $1, 2, \ldots, n$ in a fixed orientation and internal vertices coloured black and white. For a reduced plabic graph $\mathcal{G}$ corresponding to the top cell in the TNN-Grassmannian $(\mathrm{Gr}_{n-k,n}(\mathbb{R}))_{\geq 0}$, Rietsch and Williams [10] constructed a family of polytopes for positive integers r as Newton–Okounkov bodies [5,7] associated with the line bundle $r \in \mathbb{Z} \cong \mathrm{Pic}(\mathrm{Gr}_{n-k,n}(\mathbb{C}))$.

When the plabic graph $\mathcal{G} := \mathcal{G}^{\mathrm{rec}}_{k,n}$ is chosen as in [10] (see Section 4.2), the corresponding Newton–Okounkov body $\mathrm{NO}_{\mathcal{G}}$ is unimodularly equivalent to the Gelfand–Tsetlin polytope $\mathrm{GT}^1_{n-k,n}$.

The Newton–Okounkov body is by definition a closed convex hull of points; even when it is a polytope, to read off its defining inequalities is a hard problem. In [10], the authors used mirror symmetry of Grassmannians to obtain these inequalities from the tropicalization of the super-potential on an open set of the mirror Grassmannian arising from the Landau–Ginzburg model. By applying this symmetry, they give explicit defining inequalities of $\mathrm{NO}_{\mathcal{G}}$.

Lattice points in Gelfand–Tsetlin polytopes parametrize the bases of finite-dimensional irreducible representations of the Lie algebra $\mathfrak{sl}_n$. Motivated by a conjecture of Vinberg, another family of polytopes, called FFLV polytopes, is found by Feigin,

E-mail addresses: xinfang.math@gmail.com (X. Fang), fourier@math.uni-hannover.de (G. Fourier).

https://doi.org/10.1016/j.crma.2018.05.003

Please cite this article in press as: X. Fang, G. Fourier, Symmetries on plabic graphs and associated polytopes, C. R. Acad. Sci. Paris, Ser. I (2018), https://doi.org/10.1016/j.crma.2018.05.003

the second author, and Littelmann [3], whose lattice points also parametrize the bases of finite- dimensional irreducible representations of $\mathfrak{sl}_n$.

For a plabic graph $\mathcal{G}$, its mirror $\mathcal{G}^\vee$ is defined by swapping the black/white colouring of internal vertices in $\mathcal{G}$. When the plabic graph $\mathcal{G}$ corresponds to the top cell in $(\mathrm{Gr}_{n-k,n}(\mathbb{R}))_{\geq 0}$, $\mathcal{G}^\vee$ parametrizes the top cell in $(\mathrm{Gr}_{k,n}(\mathbb{R}))_{\geq 0}$.

Theorem 1. *The Newton–Okounkov body $NO_{\mathcal{G}^\vee}$ is unimodularly equivalent to* $\mathrm{FFLV}^1_{k,n}$ *(see Section 4.1 for definition).*

Another way to relate Gelfand–Tsetlin polytopes to FFLV polytopes is via a connection between the corresponding clusters in different cluster algebras. Each reduced plabic graph $\mathcal{G}$ gives a cluster $\mathcal{C}$ consisting of Plücker coordinates $\Delta_{I_1}, \dots, \Delta_{I_m}$ where $I_1, \dots, I_m$ are some $(n-k)$-element subsets of $[n] = \{1, 2, \dots, n\}$.

For $I \subset [n]$, let I^c denote its complement. Then the set $\mathcal{C}' = \{\Delta_{I_1^c}, \dots, \Delta_{I_m^c}\}$ is a cluster for $\mathrm{Gr}_{k,n}(\mathbb{C})$, corresponding to a plabic graph $\mathcal{G}^\vee$.

Corollary 1. *The Newton–Okounkov body $NO_{\mathcal{G}^\vee}$ is unimodularly equivalent to* $\mathrm{FFLV}^1_{k,n}$.

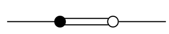
2. Plabic graphs

We recall the definition and basic properties of plabic graphs, following [8,10].

Definition 1. A *plabic graph* is an undirected planar graph $\mathcal{G}$ satisfying:

(1) $\mathcal{G}$ is embedded in a closed disk and considered up to homotopy;
(2) $\mathcal{G}$ has n vertices on the boundary of the disk, called *boundary vertices*, which are labelled clockwise by $1, 2, \dots, n$;
(3) all other vertices of $\mathcal{G}$ are strictly inside the disk, they are called *internal vertices* and coloured in black and white;
(4) each boundary vertex is incident to a single edge.

In [8] (see also [10]), there are three *local moves* defined on plabic graphs: gluing two vertices of the same colour, removing redundant vertices, and mutating a square. For a plabic graph $\mathcal{G}$, let $\mathcal{F}(\mathcal{G})$ denote the set of its faces, which is invariant under the local moves.

Definition 2. A plabic graph $\mathcal{G}$ is called *reduced* if there are no parallel edges  after applying any sequences of local moves.

Definition 3. Let $\mathcal{G}$ be a reduced plabic graph. The *trip* T_i starting from a boundary vertex i is the path going through the edges of $\mathcal{G}$, obeying the following rules:

(1) at each internal black vertex, the path turns to the rightmost direction;
(2) at each internal white vertex, the path turns to the leftmost direction.

The trip T_i ends at a boundary vertex $\pi(i)$. We associate in this way a *trip permutation* $\pi_{\mathcal{G}} := (\pi(1), \dots, \pi(n))$ with $\mathcal{G}$. Let $\pi_{k,n} = (n-k+1, n-k+2, \dots, n, 1, 2, \dots, n-k)$. The *face labelling* of $\mathcal{G}$ is the injective map $\lambda_{\mathcal{G}} : \mathcal{F}(\mathcal{G}) \to \binom{[n]}{k}$ (the set of k-element subsets of $\{1, \dots, n\}$) defined as follows: for a face $F \in \mathcal{F}(\mathcal{G})$, $\lambda_{\mathcal{G}}(F)$ consists of those i such that F is to the left of the trip T_i. We set $\mathcal{V}_{\mathcal{G}} := \lambda_{\mathcal{G}}(\mathcal{F}(\mathcal{G}))$.

See Fig. 1 for an example.

3. Polytopes arising from plabic graphs

We associate polytopes with plabic graphs following [10]. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be the base field.

3.1. Positive Grassmannians

For $0 < k < n$, let $\mathrm{Mat}_{k,n}$ denote the set of $k \times n$-matrices with entries in $\mathbb{K}$. For $J \in \binom{[n]}{k}$ and $A \in \mathrm{Mat}_{k,n}$, let $\Delta_J(A)$ denote the maximal minor of A corresponding to columns in J.

Let $\mathrm{Gr}_{k,n}$ be the Grassmann variety embedded into $\mathbb{P}^{N-1}$ via the Plücker embedding where $N = \binom{n}{k}$. The minors $\{\Delta_J \mid J \in \binom{[n]}{k}\}$ give the Plücker coordinates on $\mathrm{Gr}_{k,n}$. When the base field is $\mathbb{R}$, the *totally non-negative (resp. totally positive) Grassmannian* $(\mathrm{Gr}_{k,n}(\mathbb{R}))_{\geq 0}$ consists of those elements in $\mathrm{Gr}_{k,n}$ having non-negative (resp. positive) Plücker coordinates.

Please cite this article in press as: X. Fang, G. Fourier, Symmetries on plabic graphs and associated polytopes, C. R. Acad. Sci. Paris, Ser. I (2018), https://doi.org/10.1016/j.crma.2018.05.003