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Homological algebra/Differential geometry

Deformation cohomology of Lie algebroids and Morita equivalence

Cohomologie de déformation d'un algèbroïde de Lie et équivalence de Morita

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A R T I C L E I N F O

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A B S T R A C T

Let $A \Rightarrow M$ be a Lie algebroid. In this short note, we prove that a pull-back of A along a fibration with homologically m-connected fibers shares the same deformation cohomology of A up to degree m.

R É S U M É

Soit $A \Rightarrow M$ un algébroïde de Lie. Dans cette note, nous prouvons qu'un *pull-back* de A lelong d'une fibration ayant des fibres homologiquement m-connexes possède la même cohomologie de déformation que A jusqu'au degré m.

0. Introduction

Lie groupoids can be understood as atlases on certain singular spaces, specifically, *differentiable stacks*, and (by the very definition of stack) two Lie groupoids are Morita equivalent if they give rise to the same differentiable stack [2]. This means that, when using Lie groupoids to model differentiable stacks, Morita invariants describe the intrinsic geometry of the stack. For instance, Lie groupoid cohomology, and the deformation cohomology of a Lie groupoid are Morita invariants, but there are more many examples. The terminology is motivated by the fact that the relationship between a Lie groupoid and its stack is analogous to the relationship between a ring and its category of modules.

Lie algebroids are infinitesimal counterparts of Lie groupoids. However, the former are more general than the latter in the sense that, while all Lie groupoids *differentiate* to a Lie algebroid, not all Lie algebroids *integrate* to a Lie groupoid. A consequence of this is that there is not a notion of Morita equivalence of Lie algebroids which is universally good, but there are several non-equivalent alternatives. The weakest (but reasonable) possible one is the *weak Morita equivalence* introduced

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by Ginzburg in [9]. For Poisson manifolds, this notion is weaker than Xu's Morita equivalence [14], but it makes sense for non-integrable Poisson manifolds.

In a similar way as for Lie groupoids, it is important to identify as many Morita invariants of Lie algebroids as possible. In [4], Crainic proves (a statement equivalent to the fact) that if two Lie algebroids are Morita equivalent in a suitable sense, then they share the same de Rham cohomology in low degree. In this note, we prove the analogous result for the deformation cohomology of Lie algebroids. Notice that, for Lie groupoids, the Morita invariance of Lie groupoid cohomology has been proved by Crainic himself in [4], while the deformation cohomology has been introduced, and its Morita invariance has been proved, only very recently, by Crainic, Mestre, and Struchiner in [5].

We assume that the reader is familiar with Lie algebroids and their description in terms of graded manifolds. We only recall that a degree k $\mathbb{N}$-manifold is a graded manifold whose coordinates are concentrated in non-negative degree up to degree k, and an $\mathbb{N}Q$-manifold is an $\mathbb{N}$-manifold equipped with an homological vector field. For instance, if $A \Rightarrow M$ is a Lie algebroid, then shifting by one the degree of the fibers of the vector bundle $A \to M$, we get a degree 1 $\mathbb{N}Q$-manifold whose homological vector field is the de Rham differential d_A of A. Correspondence $A \rightsquigarrow A[1]$ establishes an equivalence between the category of Lie algebroids and the category of degree-1 $\mathbb{N}Q$-manifolds.

1. The deformation complex of a Lie algebroid

Let $A \Rightarrow M$ be a Lie algebroid. In degree k, the deformation complex of A, denoted by $(C_{\mathrm{def}}(A), \delta)$, consists of $(k+1)$-multiderivations of A, i.e. $\mathbb{R}$-$(k+1)$-linear maps

$$c : \Gamma(A) \times \cdots \times \Gamma(A) \to \Gamma(A)$$

such that there exists a (necessarily unique) vector bundle map $s_c : \wedge^k A \to TM$ with c and s_c satisfying the following Leibniz rule

$$c(\alpha_1, \ldots, \alpha_k, f\alpha_{k+1}) = s_c(\alpha_1, \ldots, \alpha_k)(f)\alpha_{k+1} + f c(\alpha_1, \ldots, \alpha_k, \alpha_{k+1}),$$

for all $\alpha_1, \ldots, \alpha_{k+1} \in \Gamma(A)$, and $f \in C^\infty(M)$. The differential δ is then defined as

$$\delta c(\alpha_0, \ldots, \alpha_{k+1}) = \sum_i (-)^i [\alpha_i, c(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_{k+1})] + \sum_{i<j} (-)^{i+j} c([\alpha_i, \alpha_j], \alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_{k+1}),$$

for all $\alpha_0, \ldots, \alpha_{k+1} \in \Gamma(A)$. The cohomology of $C_{\mathrm{def}}(A)$ is called the *deformation cohomology* of A and it is denoted by $H_{\mathrm{def}}(A)$ [7].

Actually, $C_{\mathrm{def}}(A)$ is not just a complex, but it is the DG Lie algebra (even more a DG Lie–Rinehart algebra over the de Rham algebra of A) controlling deformations of A, in the sense that

- ▹ Lie algebroid structures on A corresponds bijectively to Maurer–Cartan elements in $C_{\mathrm{def}}(A)$, and
- ▹ if two Lie algebroid structures are isotopic, the corresponding Maurer–Cartan elements are gauge equivalent, and the converse is also true when M is compact.

There is a simple alternative description of $C_{\mathrm{def}}(A)$ as the DG Lie algebra of graded derivations of the de Rham algebra $(C(A), \mathrm{d}_A)$, where $C(A) = \Gamma(\wedge^\bullet A^*)$, and d_A is the usual Lie algebroid de Rham differential. A cochain $c \in C^k_{\mathrm{def}}(A)$ corresponds to the degree k derivation D_c mapping $\omega \in C^l(A)$ to $D_c\omega \in C^{k+l}(A)$, with

$$D_c\omega(\alpha_1, \ldots, \alpha_{k+l}) = \sum_{\sigma \in S_{k,l}} (-)^\sigma s_c(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)})(\omega(\alpha_{\sigma(k+1)}, \ldots \alpha_{\sigma(k+l)})) - \sum_{\sigma \in S_{k+1,l-1}} (-)^\sigma \omega(c(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+1)}), \alpha_{\sigma(k+2)}, \ldots \alpha_{\sigma(k+l)}).$$

When taking this point of view, the Lie bracket in $C_{\mathrm{def}}(A)$ is just the graded commutator $[-,-]$ of derivations and δ is $[\mathrm{d}_A, -]$. Finally, we can interpret $C(A)$ as the DG algebra of smooth functions on the degree 1 $\mathbb{N}Q$-manifold $A[1]$, and then cochains in $C_{\mathrm{def}}(A)$ are just vector fields on $A[1]$:

$$C(A) = C^\infty(A[1]), \quad \text{and} \quad C_{\mathrm{def}}(A) = \mathfrak{X}(A[1]).$$

In the following, we will mostly take this point of view.

Given two Lie algebroids $A \Rightarrow M$ and $B \Rightarrow N$ and a Lie algebroid map $F : A \to B$ covering a smooth map $M \to N$, there is a DG algebra map $F^* : C(B) \to C(A)$. One can also connect the deformation complexes as follows. Apply the shift functor to