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Algebraic geometry

Deformation of the product of complex Fano manifolds

Déformation du produit de variétés de Fano complexes

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A R T I C L E I N F O

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A B S T R A C T

Let $\mathcal{X}$ be a connected family of complex Fano manifolds. We show that if some fiber is the product of two manifolds of lower dimensions, then so is every fiber. Combining with previous work of Hwang and Mok, this implies immediately that if a fiber is a (possibly reducible) Hermitian symmetric space of compact type, then all fibers are isomorphic to the same variety.

R É S U M É

Soit $\mathcal{X}$ une famille connexe des variétés de Fano complexes. On montre que, si une fibre est un produit de deux variétés de dimensions inférieures, alors il en est de même pour chaque fibre. En combinant avec un résultat de Hwang et Mok, ceci implique immédiatement que, si une fibre est un espace Hermitien symétrique de type compact, alors toutes les fibres sont isomorphes à cette variété.

We consider varieties defined over the field of complex numbers $\mathbb{C}$. The aim of this short note is to show the following result.

Theorem 1. *Let $\pi : \mathcal{X} \to S \ni 0$ be a holomorphic map onto a connected complex manifold S such that all fibers are connected Fano manifolds and $\mathcal{X}_0 \cong \mathcal{X}'_0 \times \mathcal{X}''_0$. Then there exist unique holomorphic maps*

$$f' : \mathcal{X} \to \mathcal{X}' \ \pi' : \mathcal{X}' \to S,$$
$$f'' : \mathcal{X} \to \mathcal{X}'' \ \pi'' : \mathcal{X}'' \to S,$$

such that $\mathcal{X}'_0$ and $\mathcal{X}''_0$ coincide with the fibers of π' and π'' at $0 \in S$, the morphism $\pi = \pi' \circ f' = \pi'' \circ f''$ and $\mathcal{X}_t \cong \mathcal{X}'_t \times \mathcal{X}''_t$ for all $t \in S$.

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It is necessary to assume that each fiber $\mathcal{X}_t$ is Fano in Theorem 1. For example, the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ can be deformed to a Hirzebruch surface Σ_m with m even. On the other hand, a Hirzebruch surface Σ_m with $m \neq 0$ is neither a Fano manifold nor a product of two curves.

The following vanishing result is standard, which helps to study relative Mori contractions in our setting. The latter is important for our proof of Theorem 1.

Proposition 2. *Let $\pi : \mathcal{X} \to \Delta^n$ be a holomorphic map onto the unit disk Δ^n of dimension n such that all fibers are connected Fano manifolds. Suppose that $\mathcal{L}$ is a holomorphic line bundle on $\mathcal{X}$ such that $\mathcal{L}_t := \mathcal{L}|_{\mathcal{X}_t}$ is nef on $\mathcal{X}_t$ for each $t \in \Delta^n$. Then $H^k(\mathcal{X}, \mathcal{L}) = 0$ for all $k \geq 1$.*

Proof. Since each $\mathcal{X}_t$ is Fano, $H^k(\mathcal{X}_t, \mathcal{L}_t) = 0$ and $R^k\pi_*\mathcal{L} = 0$ for all $k \geq 1$ by Kodaira vanishing. Then $H^k(\mathcal{X}, \mathcal{L}) = H^k(\Delta^n, \pi_*\mathcal{L}) = 0$ for all $k \geq 1$ since Δ^n is a Stein manifold. □

We summarize several facts in Proposition 3 on the relative Mori contraction in our setting. One can consult [3] or [5] for standard notations and results in relative Minimal Model Program. By [7, Theorem 1], the Mori cone of each $\mathcal{X}_t$ is invariant in our setting, which we restate as conclusion (ii) in Proposition 3.

Proposition 3. *Let $\pi : \mathcal{X} \to \Delta^n$ be as in Proposition 2. Then, for each $t \in \Delta^n$, the following holds.*

(i) The natural morphism $Pic(\mathcal{X}) \to Pic(\mathcal{X}_t)$ is an isomorphism.

(ii) There are natural identities as follows:

$$N_1(\mathcal{X}_t) = N_1(\mathcal{X}/\Delta^n) \text{ and } \overline{NE}(\mathcal{X}_t) = \overline{NE}(\mathcal{X}/\Delta^n). \tag{1}$$

(iii) Denote by $\Phi : \mathcal{X} \to \mathcal{Y}$ the relative Mori contraction over Δ^n associated with an extremal face $F \subset \overline{NE}(\mathcal{X}/\Delta^n)$. Then Φ_t is the Mori contraction associated with $F \subset \overline{NE}(\mathcal{X}_t)$ under identification (1). In particular, $\mathcal{Y}$ and $\mathcal{Y}_t$ are normal varieties.

Proof. (i) By Proposition 2, the map $e : \mathrm{Pic}(\mathcal{X}) \to H^2(\mathcal{X}, \mathbb{Z})$, induced by exponential sheaf sequence, is an isomorphism. Since $\mathcal{X}_t$ is a Fano manifold, the map $e_t : \mathrm{Pic}(\mathcal{X}_t) \to H^2(\mathcal{X}_t, \mathbb{Z})$ is an isomorphism. The natural map $r : H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}_t, \mathbb{Z})$ is also an isomorphism, since Δ^n is a contractible topology space. Then (i) follows from the fact that the natural morphism $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_t)$ coincides with the composition $e_t^{-1} \circ r \circ e$.

(ii) The linear map $N^1(\mathcal{X}/\Delta^n) \to N^1(\mathcal{X}_t)$ is surjective by (i). Then we have injective linear maps

$$\gamma_t : N_1(\mathcal{X}_t) \to N_1(\mathcal{X}/\Delta^n) \text{ and } \gamma_t^+ : \overline{NE(\mathcal{X}_t)} \to \overline{NE}(\mathcal{X}/\Delta^n).$$

By Theorem 1 in [7], the local identification $H_2(\mathcal{X}_0, \mathbb{R}) = H_2(\mathcal{X}_t, \mathbb{R})$ yields the identity of Mori cones $\overline{NE}(\mathcal{X}_0) = \overline{NE}(\mathcal{X}_t)$. Since the local identification $H^2(\mathcal{X}_0, \mathbb{R}) = H^2(\mathcal{X}_t, \mathbb{R})$ gives the identity of $\mathcal{L}|_{\mathcal{X}_0}$ and $\mathcal{L}|_{\mathcal{X}_t}$ for any $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$, we know by (i) that

$$\gamma_t^+(\overline{NE}(\mathcal{X}_t)) = \gamma_0^+(\overline{NE}(\mathcal{X}_0)),\ \gamma_t(N_1(\mathcal{X}_t)) = \gamma_0(N_1(\mathcal{X}_0)). \tag{2}$$

The abelian group $Z_1(\mathcal{X}/\Delta^n)$ (resp. semigroup $Z_1^+(\mathcal{X}/\Delta^n)$), generated by reduced irreducible curves that are contracted by π, is isomorphic to $\bigoplus_{s \in \Delta^n} Z_1(\mathcal{X}_s)$ (resp. $\bigoplus_{s \in \Delta^n} Z_1^+(\mathcal{X}_s)$). Then (2) implies that γ_t and γ_t^+ are bijections.

(iii) By the Contraction Theorem, there exist $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ and a homomorphism $\Phi : \mathcal{X} \to \mathcal{Y} := \mathrm{Proj}_{\Delta^n} R(\mathcal{L})$ over Δ^n such that $\Phi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$ and

$$F = \{\eta \in \overline{NE}(\mathcal{X}/\Delta^n) \mid \deg \mathcal{L}|_\eta = 0\},$$

where $R(\mathcal{L}) := \bigoplus_{m \geq 0} H^0(\mathcal{X}, \mathcal{L}^{\otimes m})$. Then (iii) follows from (ii) and the claim that $H^0(\mathcal{X}, \mathcal{L}^{\otimes m}) \to H^0(\mathcal{X}_t, \mathcal{L}_t^{\otimes m})$ is surjective for all $m \geq 0$.

The Cartier divisor $\mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{X}_t)$ is nef on $\mathcal{X}_s$ for all $s \in \Delta^n$, which implies that $H^1(\mathcal{X}, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{X}_t)) = 0$ by Proposition 2. Hence, the homomorphism $H^0(\mathcal{X}, \mathcal{L}^{\otimes m}) \to H^0(\mathcal{X}_t, \mathcal{L}_t^{\otimes m})$, induced by the short exact sequence

$$0 \to \mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{X}_t) \to \mathcal{L}^{\otimes m} \to \mathcal{L}_t^{\otimes m} \to 0,$$

is surjective, which proves the claim. □

Local verification of Theorem 1 is closely related to Kodaira's stability as follows.