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Group theory

# Large subgroups in finite groups

## *Grands sous-groupes dans les groupes finis*

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A B S T R A C T

Following Isaacs (see [6, p. 94]), we call a normal subgroup $N$ of a finite group $G$ *large*, if $\mathbf{C}_G(N) \le N$, so that $N$ has bounded index in $G$. Our principal aim here is to establish some general results for systematically producing large subgroups in finite groups (see Theorems A and C). We also consider the more specialised problems of finding large (non-abelian) nilpotent as well as abelian subgroups in soluble groups.


R É S U M É

Suivant la terminologie introduite par Isaacs (voir [6], p. 94), nous disons qu'un sous-groupe distingué $N$ d'un groupe fini $G$ est *grand* si $\mathbf{C}_G(N) \le N$, de sorte que $N$ est d'indice borné dans $G$. Notre but principal est d'établir des résultats permettant de produire de façon systématique des grands sous-groupes dans les groupes finis (voir les théorèmes A et C). Nous considérons également les problèmes plus particuliers qui se posent pour trouver de grands sous-groupes nilpotents (non commutatifs) ainsi que de grands sous-groupes commutatifs dans les groupes résolubles.


## 1. Introduction

Let $G$ be a finite group and suppose that $N$ is a normal subgroup of $G$ such that $\mathbf{C}_G(N) \le N$. Then we call $N$ a *large* subgroup of $G$. The motivation for naming such a subgroup "large" stems from the observation that if $N \trianglelefteq G$, then the factor group $G/\mathbf{C}_G(N)$ is isomorphically embedded in $\mathbf{Aut}(N)$, and so $|G:\mathbf{C}_G(N)| \le |\mathbf{Aut}(N)|$. Thus, if $N$ is large, it follows that $|G:N| \le |\mathbf{Aut}(N)|$, so that $N$ has bounded index in $G$.

There are a number of examples of large subgroups in the literature. For instance, if $G$ is soluble, then the (standard) Fitting subgroup $\mathbf{F}(G)$ of $G$ is large. Also, if $G$ is $\pi$-separable and $\mathbf{O}_{\pi'}(G) = 1$, then $\mathbf{O}_\pi(G)$ is large (this result is known as the Hall–Higman Lemma 1.2.3). Moreover, if $G$ is any finite group, then the generalised Fitting subgroup $\mathbf{F}^*(G)$ of $G$ is large.

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We briefly recall the definition of Bender's subgroup $\mathbf{F}^*(G)$. Call a group $G$ *quasi-simple*, if $G$ is perfect and $G/\mathbf{Z}(G)$ is simple. A subnormal quasi-simple subgroup of an arbitrary group $G$ is called a *component* of $G$ and the *layer* of $G$, denoted by $\mathbf{E}(G)$, is the product of its components. It is known that, if $A$ and $B$ are components of $G$, then either $A = B$ or $[A, B] = 1$. The *generalised Fitting subgroup* of $G$, denoted by $\mathbf{F}^*(G)$, is the product of $\mathbf{E}(G)$ and $\mathbf{F}(G)$.

We shall reserve fraktur symbols for classes of finite groups; in particular, $\mathfrak{S}$ will denote the class of finite soluble groups, $\mathfrak{N}$ the class of finite nilpotent groups, and $\mathfrak{A}$ the class of finite abelian groups.

A formation $\mathfrak{F}$ with the property that $G/\mathbf{\Phi}(G) \in \mathfrak{F}$ for $G$ finite implies $G \in \mathfrak{F}$ is called saturated. We recall that if $\mathfrak{F}$ is a formation, then $\mathfrak{F}$ is called *solubly saturated* provided that $G \in \mathfrak{F}$ whenever $G/\mathbf{\Phi}(G_{\mathfrak{S}}) \in \mathfrak{F}$, where $G_{\mathfrak{S}}$ is the soluble radical of $G$; that is, the largest soluble normal subgroup of $G$. If $\mathfrak{X}$ is a class of finite groups, then we shall write $\overline{\mathfrak{X}}$ for its extension closure; that is, the class of all (finite) groups having a subnormal series whose quotients lie in $\mathfrak{X}$.

## 2. Main results

Before establishing our main theorems (see Theorems A and C below), we need to briefly discuss some auxiliary results.

**Lemma 1.** *Let $\mathfrak{X}$ be a class of finite groups which is closed under taking normal subgroups and quotients. Then $\overline{\mathfrak{X}}$, the extension closure of $\mathfrak{X}$ (i.e. the class of poly-$\mathfrak{X}$ groups), enjoys the same closure properties.*

**Proof.** If $\mathfrak{X}$ is quotient-closed then so is $\overline{\mathfrak{X}}$ by Part (vi) of [1, Prop. 5.4]. Also, a straightforward modification of the proof of Part (vii) of [1, Prop. 5.4] shows that $\overline{\mathfrak{X}}$ inherits normal-subgroup-closure from $\mathfrak{X}$. □

**Lemma 2.** *Let $\mathfrak{X}$ be a normal-subgroup-closed class of finite groups and let $\overline{\mathfrak{X}}$ be its extension closure. If $G \in \overline{\mathfrak{X}}$ and $J$ is a simple subnormal subgroup of $G$, then $J \in \mathfrak{X}$.*

**Proof.** Since $\mathfrak{X}$ is closed under taking normal subgroups, a subnormal series for $G$ with quotients in $\mathfrak{X}$ can be refined to yield a composition series with the same property. Since $J$ can be made the first term of a composition series for $G$, the result follows by the Jordan–Hölder theorem. □

**Corollary 3.** *Let $\mathfrak{X}$ be a class of finite groups which is closed under taking normal subgroups and direct products. If $G \in \overline{\mathfrak{X}}$ and $N$ is a minimal normal subgroup of $G$, then $N \in \mathfrak{X}$.*

**Proof.** Since $N$ is a minimal normal subgroup of $G$, we have that $N \cong J \times \cdots \times J$ for some simple group $J$. In particular, $J$ is subnormal in $G$ and thus Lemma 2 applies to yield that $J \in \mathfrak{X}$. Now direct-product-closure of $\mathfrak{X}$ shows that $N \in \mathfrak{X}$. □

We are now in a position to prove our first main result.

**Theorem A.** *Let $\mathfrak{X}$ be a class of finite groups which is closed under taking normal subgroups, direct products, quotients, and central extensions, and let $\overline{\mathfrak{X}}$ be the extension closure of $\mathfrak{X}$. Suppose that $G \in \overline{\mathfrak{X}}$, and that $H$ is a maximal normal $\mathfrak{X}$-subgroup of $G$. Then $H$ is large in $G$.*

**Proof.** Let $G \in \overline{\mathfrak{X}}$, and let $H$ be a maximal normal $\mathfrak{X}$-subgroup of $G$. We claim that $H$ is large in $G$ or, equivalently, that $\mathbf{Z}(H) = \mathbf{C}_G(H)$. Assume for a contradiction that $C > Z$, where $C := \mathbf{C}_G(H)$ and $Z := \mathbf{Z}(H)$.

Since $H$ is a normal subgroup of $G$, so are $C$ and $Z$, and thus $C/Z$ is a non-trivial normal subgroup of $G/Z$. Note that $G \in \overline{\mathfrak{X}}$ implies that $G/Z \in \overline{\mathfrak{X}}$ by Lemma 1. Also, $C/Z \in \overline{\mathfrak{X}}$, again by Lemma 1. Now put $D/Z := \mathbf{Soc}(C/Z)$. By Corollary 3, we have $D/Z \in \mathfrak{X}$ and $1 < D/Z$ since $C/Z > 1$ by hypothesis. Also, $H/Z \in \mathfrak{X}$, since $\mathfrak{X}$ is quotient-closed. Now note that $(H/Z) \cap (D/Z) = 1$ since $Z \le H \cap D \le H \cap C = Z$. Since $\mathfrak{X}$ is closed under taking direct products, it follows that $HD/Z = (H/Z) \times (D/Z) \in \mathfrak{X}$, thus central-extension-closure of $\mathfrak{X}$ yields $HD \in \mathfrak{X}$. However, $HD > H$, and moreover $HD$ is a normal subgroup of $G$, which contradicts the maximality of $H$. □

Our next result guarantees the existence of a variety of natural classes of finite groups that are central-extension-closed.

**Proposition B.** *If $\mathfrak{X}$ is a solubly saturated formation of finite groups that is closed under taking normal subgroups and such that $\mathfrak{A} \subseteq \mathfrak{X}$, then $\mathfrak{X}$ is closed under taking central extensions.*

**Proof.** Let $G$ be a finite group, $1 < Z \le \mathbf{Z}(G)$, and suppose that $G/Z \in \mathfrak{X}$, with $\mathfrak{X}$ a class of finite groups as in the statement of the proposition.

First, we argue that, since $Z$ is an abelian group, we can find an abelian group $Y$ such that $\mathbf{\Phi}(Y) \cong Z$. This follows from the structure theorem for abelian groups, the fact that $\mathbf{\Phi}(C_{p^{\alpha+1}}) \cong C_{p^\alpha}$, and [4, Satz 6].

Now let $\Gamma = G \circ Y$ be the central product of $G$ with $Y$, identifying $Z$ with the Frattini subgroup of $Y$. Then $\Gamma/Z \cong (G/Z) \times (Y/Z)$, so $\Gamma/Z \in \mathfrak{X}$ since $Y/Z \in \mathfrak{A} \subseteq \mathfrak{X}$ and $\mathfrak{X}$ is a formation thus closed under taking direct products. Also, $Z \le$

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