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Mathematical problems in mechanics

The asymptotically sharp Korn interpolation and second inequalities for shells

Inégalité d'interpolation et seconde inégalité de Korn asymptotiquement fines pour les coques

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A R T I C L E   I N F O

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A B S T R A C T

We consider shells in three-dimensional Euclidean space that have bounded principal curvatures. We prove Korn's interpolation (or the so-called first and a half[1]) and the second inequalities on that kind of shells for $\boldsymbol{u} \in W^{1,2}$ vector fields, imposing no boundary or normalization conditions on $\boldsymbol{u}$. The constants in the estimates are optimal in terms of the asymptotics in the shell thickness h, having the scalings h or $O(1)$. The Korn interpolation inequality reduces the problem of deriving any linear Korn type estimate for shells to simply proving a Poincaré-type estimate with the symmetrized gradient on the right-hand side. In particular, this applies to linear geometric rigidity estimates for shells, i.e. Korn's fist inequality without boundary conditions.

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R É S U M É

Nous considérons les coques dans un espace euclidien de dimension trois, dont la courbure principale est bornée. Nous établissons l'inégalité d'interpolation de Korn (aussi nommée première et demi) et la seconde inégalité de Korn, sur ce type de coque, pour un champ de vecteurs $\boldsymbol{u} \in W^{1,2}$, sans imposer de borne ou de condition de normalisation à $\boldsymbol{u}$. Les constantes des estimations sont optimales en termes d'asymptotique en l'épaisseur h de la coque avec les échelles h ou $O(1)$. L'inégalité d'interpolation de Korn est plus forte que la classique seconde inégalité de Korn, et il apparaît qu'elle est précise pour tous les types de courbure principaux (zéro, positive, négative). Ainsi, cette précision réduit le problème de l'obtention de n'importe quelle estimation linéaire de type Korn pour les coques à simplement démontrer une estimation de type Poincaré avec le gradient symétrisé dans le membre de droite. En particulier, ceci s'applique aux estimations de rigidité géométrique linéaires pour les coques, c'est-à-dire la première inégalité de Korn sans condition de bord.

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[1] The inequality first introduced in [6].

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1. Introduction

A shell of thickness h in three-dimensional Euclidean space is given by $\Omega=\{x+t\boldsymbol{n}(x)\ :\ x\in S,\ t\in[-h/2,h/2]\}$, where $S\subset\mathbb{R}^3$ is a bounded and connected smooth enough regular surface with a unit normal $\boldsymbol{n}(x)$ at the point $x\in S$. The surface S is called the mid-surface of the shell Ω. Understanding the rigidity of a shell is one of the challenges in nonlinear elasticity, where there are still many open questions. Unlike the situation for shells in general, the rigidity of plates has been quite well understood by Friesecke, James and Müller in their celebrated papers [4,5]. It is known that the rigidity of a shell Ω is closely related to the optimal Korn constant in the nonlinear (in some cases linear) first Korn inequality [5,6], which is a geometric rigidity estimate for $\boldsymbol{u}\in H^1(\Omega)$ fields [9,4,1–3]. Depending on the problem, the field $\boldsymbol{u}\in H^1$ may or may not satisfy boundary conditions, e.g., [9,5,6]. Finding the optimal constants in Korn's inequalities is a central task in problems concerning shells in general. The Friesecke–James–Müller estimate reads as follows. *Assume $\Omega\subset\mathbb{R}^3$ is open bounded connected and Lipschitz. Then there exists a constant $C_I=C_I(\Omega)$, such that, for every vector field $\boldsymbol{u}\in H^1(\Omega)$, there exists a constant rotation $\boldsymbol{R}\in SO(3)$, such that*

$$\|\nabla\boldsymbol{u}-\boldsymbol{R}\|^2\leq C_I\int_\Omega \mathrm{dist}^2(\nabla\boldsymbol{u}(x),SO(3))\,\mathrm{d}x. \tag{1.1}$$

The linearization of (1.1) around the identity matrix is Korn's first inequality [11,12,10,4,1] without boundary conditions and reads as follows. *Assume $\Omega\subset\mathbb{R}^n$ is open bounded connected and Lipschitz. Then there exists a constant $C_{II}=C_{II}(\Omega)$, depending only on Ω, such that for every vector field $\boldsymbol{u}\in H^1(\Omega)$, there exists a skew-symmetric matrix $\boldsymbol{A}\in\mathbb{R}^{n\times n}$, i.e. $A+A^T=0$, such that*

$$\|\nabla\boldsymbol{u}-\boldsymbol{A}\|^2_{L^2(\Omega)}\leq C_{II}\|e(\boldsymbol{u})\|^2_{L^2(\Omega)}, \tag{1.2}$$

where $e(\boldsymbol{u})=\frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$ is the symmetrized gradient (the strain in linear elasticity). The estimate (1.2) is traditionally proven by using Korn's second inequality, that reads as follows: *Assume $\Omega\subset\mathbb{R}^n$ is open bounded connected and Lipschitz. Then there exists a constant $C=C(\Omega)$, depending only on Ω, such that for every vector field $\boldsymbol{u}\in H^1(\Omega)$ there holds:*

$$\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)}\leq C(\|\boldsymbol{u}\|^2_{L^2(\Omega)}+\|e(\boldsymbol{u})\|^2_{L^2(\Omega)}). \tag{1.3}$$

It is known that if Ω is a thin domain with thickness h, then in general the optimal constants C in all inequalities (1.1)–(1.3) blow up as $h\to 0$. In particular, if Ω is a plate given by $\Omega=\omega\times(0,h)$, where $\omega\subset\mathbb{R}^2$ is open bounded connected and Lipschitz, then, as proven in [5], one has $C_I=c_1(\omega)h^2$ and $C_{II}=c_2(\omega)h^2$ asymptotically as $h\to 0$. While the asymptotics of C_{II} is known in the case when $\boldsymbol{u}$ satisfies zero Dirichlet boundary conditions on the thin face of the shell [7,8] (C_{II} scaling like h^2, $h^{3/2}$, $h^{4/3}$ or h^1), it is open for general fields $\boldsymbol{u}\in H^1(\Omega)$. In this work, we are concerned with the asymptotics of the constant C in (1.3) or more precisely in the so-called Korn interpolation inequality, or the first-and-a-half Korn inequality [6], in the general case when Ω is a shell. The statements solving the problem practically completely appear in the next section.

2. Main results

We first introduce the main notation and definitions. We will assume throughout this work that the mid-surface S of the shell Ω is connected, compact, regular, and of class C^3 up to its boundary. We also assume that S has a finite atlas of patches $S\subset\cup_{i=1}^k\Sigma_i$ such that each patch Σ_i can be parametrized by the principal variables z and θ ($z=$ constant and $\theta=$ constant are the principal lines on Σ_i) that change in the ranges $z\in[z_i^1(\theta),z_i^2(\theta)]$ for $\theta\in[0,\omega_i]$, where $\omega_i>0$ for $i=1,2,\dots,k$. Moreover, the functions $z_i^1(\theta)$ and $z_i^2(\theta)$ satisfy the conditions

$$\begin{gathered}\min_{1\leq i\leq k}\inf_{\theta\in[0,\omega_i]}[z_i^2(\theta)-z_i^1(\theta)]=l>0,\quad \max_{1\leq i\leq k}\sup_{\theta\in[0,\omega_i]}[z_i^2(\theta)-z_i^1(\theta)]=L<\infty,\\ \max_{1\leq i\leq k}\left(\|z_i^1\|_{W^{1,\infty}[0,\omega_i]}+\|z_i^2\|_{W^{1,\infty}[0,\omega_i]}\right)=Z<\infty.\end{gathered} \tag{2.1}$$

Since there will be no condition imposed on the vector field $\boldsymbol{u}\in H^1(\Omega)$, (see Theorem 2.1), we can restrict ourselves to a single patch $\Sigma\subset S$ and denote it by S for simplicity. If the parametrization of S is $\boldsymbol{r}=\boldsymbol{r}(\theta,z)$ and $\boldsymbol{n}$ is the unit normal to S, denoting the normal variable by t and $A_z=\left|\frac{\partial\boldsymbol{r}}{\partial z}\right|$, $A_\theta=\left|\frac{\partial\boldsymbol{r}}{\partial\theta}\right|$, we get

$$\nabla\boldsymbol{u}=\begin{bmatrix} u_{t,t} & \dfrac{u_{t,\theta}-A_\theta\kappa_\theta u_\theta}{A_\theta(1+t\kappa_\theta)} & \dfrac{u_{t,z}-A_z\kappa_z u_z}{A_z(1+t\kappa_z)}\\ u_{\theta,t} & \dfrac{A_zu_{\theta,\theta}+A_zA_\theta\kappa_\theta u_t+A_{\theta,z}u_z}{A_zA_\theta(1+t\kappa_\theta)} & \dfrac{A_\theta u_{\theta,z}-A_{z,\theta}u_z}{A_zA_\theta(1+t\kappa_z)}\\ u_{z,t} & \dfrac{A_zu_{z,\theta}-A_{\theta,z}u_\theta}{A_zA_\theta(1+t\kappa_\theta)} & \dfrac{A_\theta u_{z,z}+A_zA_\theta\kappa_z u_t+A_{z,\theta}u_\theta}{A_zA_\theta(1+t\kappa_z)}\end{bmatrix} \tag{2.2}$$

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