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# Complex variable approach to the analysis of a fractional differential equation in the real line

## *Approche par variable complexe de l'analyse d'une équation différentielle fractionnaire sur la droite réelle*

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A B S T R A C T

The first aim of this work is to establish a Peano-type existence theorem for an initial value problem involving a complex fractional derivative, and then, as a consequence of this theorem, to give a partial answer for the local existence of the continuous solution to the initial value problem:

$$\begin{cases} D_x^q u(x) = f\big(x, u(x)\big), \\ u(0) = b, \quad (b \neq 0). \end{cases}$$

Moreover, for some special cases of the problem, we investigate the corresponding geometric properties of the solutions.



R É S U M É

L'objectif principal de ce travail est d'établir un théorème d'existence de type Peano pour un problème aux valeurs initiales faisant intervenir une dérivée fractionnaire, puis, comme conséquence, de donner une réponse partielle à l'existence locale d'une solution continue du problème aux valeurs initiales suivant :

$$\begin{cases} D_x^q u(x) = f\big(x, u(x)\big), \\ u(0) = b, \quad (b \neq 0). \end{cases}$$

De plus, nous étudions les propriétés géométriques des solutions pour quelques cas particuliers.



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## 1. Introduction and motivation

In the theory of ordinary differential equations, Peano's existence theorem is practical and important because one can easily check the local existence of the solution to a differential equation. The Peano-type existence theorem in the theory of fractional differential equations was first given by Laksmikantham and Vatsala [10] by using Tonelli's approach for the initial value problem

$$\begin{cases} D_x^q u(x) = f\big(x, u(x)\big), \\ u(0) = b, \end{cases} \tag{1.1}$$

where $D_x^q$ is the well-known Riemann–Liouville fractional derivative in the real line, $b$ is a real number, and $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$. However, as indicated in [24], problem (1.1) with $b \neq 0$ is not a suitable problem. For this reason, this problem was considered with different initial data or with a modified equation by some researchers, see, for example [1], [6], [9], [15], [24]. Nevertheless, if the initial condition in (1.1) is taken as homogenous, one can see that problem (1.1) is meaningful provided that $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$. In any case, problem (1.1) with the non-homogenous initial condition, that is, $b \neq 0$, can be suitable if the nonlinear function $f$ in this problem satisfies the following two conditions:

(i) $f(x,y)$ and $x^q f(x,y)$ $(0 < q < 1)$ are continuous on $(0,T] \times \mathbb{R}$ and $[0,T] \times \mathbb{R}$, respectively,
(ii) $x^q f(x,b)\big|_{x=0} = b/\Gamma(1-q)$, where $\Gamma$ is the well-known Gamma function.

It is known from the study of Delbosco and Rodino [3] that if condition (i) holds, then the equation (1.1) is equivalent to the following integral equation:

$$u\,(x) \;=\; \frac{1}{\Gamma\,(q)} \int_0^x \frac{f\,(\zeta, u\,(\zeta))}{(x-\zeta)^{1-q}}\,\mathrm{d}\zeta, \tag{1.2}$$

Now, if the initial condition with $b \neq 0$ in (1.1) is considered, then condition (ii) is necessary for the problem to be suitable. Otherwise

$$b = u(0) = \lim_{x \to 0^+} u(x) = \frac{1}{\Gamma(q)} \lim_{x \to 0^+} \int_0^1 \frac{(xt)^q\ f\,(xt, u\,(xt))}{t^q\,(1-t)^{1-q}}\,\mathrm{d}t \neq b,$$

which is a contradiction.

It is to be observed that the existence and uniqueness of continuous solutions to this problem without initial condition was proved in Theorem 3.5 in [3] by posing a Lipschitz-type condition in addition to (i). However, the local existence of continuous solutions to this problem only under condition (i) is still an open problem.

In this study, we give a partial answer to this open problem by establishing a Peano-type existence theorem for the complex version of problem (1.1):

$$\begin{cases} D_z^q u(z) = f\big(z, u(z)\big), \quad z \in \mathbb{U}_R, \\ u(0) = b, \end{cases} \tag{1.3}$$

where $\mathbb{U}_R : \big\{z \in \mathbb{C} : |z| < R\big\}$ is an open disc in the complex plane $\mathbb{C}$, $b \in \mathbb{C}$ and $D^q$ is the complex fractional derivative operator, which is a direct extension of the well-known Riemann–Liouville derivative in the real line and given in [18], [20].

The existence of a solution to problem (1.3) is investigated in the Banach space $\mathcal{B}(\mathbb{U}_R) := C(\overline{\mathbb{U}}_R) \cap \mathcal{A}$, where $\mathbb{U}_R := \big\{z \in \mathbb{C} : |z| < R\big\}$, $C(\overline{\mathbb{U}}_R)$ is the space of continuous functions on $\overline{\mathbb{U}}_R$, and $\mathcal{A}$ is the space of analytic functions on $\mathbb{U}_R$, provided that the nonlinear function $f$ satisfies the following two conditions:

(iii) $f(z,t)$ is a multivalued function on $\overline{\mathbb{U}}_R \times \mathbb{C}$ such that $z^q f(z,t)$ is analytic on $\mathbb{U}_R \times \mathbb{C}$ and continuous on $\overline{\mathbb{U}}_R \times \mathbb{C}$;
(iv) $z^q f(z,b)\big|_{z=0} = b/\Gamma(1-q)$.

By the help of this existence theorem (see Corollary 2.6 and Example 2.7), it is possible to show the existence of continuous solutions to problem (1.1) in the following way: if the nonlinear function $f(x,t)$ satisfying conditions (i)–(ii) can be extended to the function $f(z,t)$ satisfying conditions (iii)–(iv), then, as a result of this existence theorem, one can say that there exists a solution $u \in \mathcal{B}(\mathbb{U}_R)$ of (1.3). At the same time, the real part of $u \in \mathcal{B}(\mathbb{U}_R)$ is a continuous solution to problem (1.1). From this point of view, this existence theorem contributes to the area of fractional differential equations in the real line. Moreover, in the proof of the theorem, we use a new technique related to Schwarz' Lemma for analytic functions of one variable and two variables varying on generally different closed balls in $\mathbb{C}$, which we prove in the next section. From this aspect, this existence result is a nice consequence of the analytic functions.