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Partial differential equations

On parabolic final value problems and well-posedness

Sur les problèmes paraboliques à valeur finale bien posés

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A R T I C L E I N F O

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A B S T R A C T

We prove that a large class of parabolic final value problems is well posed. This results via explicit Hilbert spaces that characterise the data yielding existence, uniqueness and stability of solutions. This data space is the graph normed domain of an unbounded operator, which represents a new compatibility condition pertinent for final value problems. The framework is that of evolution equations for Lax–Milgram operators in vector distribution spaces. The final value heat equation on a smooth open set is also covered, and for non-zero Dirichlet data, a non-trivial extension of the compatibility condition is obtained by addition of an improper Bochner integral.

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R É S U M É

Nous prouvons que les problèmes à valeur finale sont bien posés pour une large classe d'opérateurs differentiels paraboliques. Ceci est obtenu via un espace de Hilbert qui caractérise l'existence des données impliquant l'existence, l'unicité et la stabilité des solutions. Cet espace de données est le domaine d'un opérateur non borné muni de la norme du graphe, qui représente une nouvelle condition de compatibilité pertinente pour les problèmes à valeur finale. Le cadre est celui des équations d'évolution pour des opérateurs de Lax–Milgram dans des espaces de distributions vectorielles. Nous étudions aussi le problème à valeur finale pour l'équation de la chaleur sur un ouvert lisse ; pour des données de Dirichlet non nulles, nous obtenons une extension non triviale de la condition de compatibilité par l'addition d'une intégrale de Bochner impropre.

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1. Introduction

Well-posedness of final value problems for a large class of parabolic differential equations is described here. That is, for suitable spaces X, Y specified below, they have *existence, uniqueness* and *stability* of solutions $u \in X$ for given data $(f, g, u_T) \in Y$. This should provide a basic clarification of a type of problems, which hitherto has been insufficiently understood.

As a first example, we characterise the functions $u(t,x)$ that, in a C^∞-smooth bounded open set $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$, satisfy the following equations that constitute the final value problem for the heat equation ($\Delta = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$ denotes the Laplacian):

$$\left.\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) &= f(t,x) && \text{for } t \in]0,T[,\ x \in \Omega, \\ u(t,x) &= g(t,x) && \text{for } t \in]0,T[,\ x \in \partial\Omega, \\ u(T,x) &= u_T(x) && \text{for } x \in \Omega. \end{aligned}\right\} \tag{1}$$

Hereby (f, g, u_T) are the given data of the problem.

In case $f = 0$, $g = 0$, the first two lines of (1) are satisfied by $u(t,x) = \mathrm{e}^{(T-t)\lambda} v(x)$ for all $t \in \mathbb{R}$, if $v(x)$ is an eigenfunction of the Dirichlet realization $-\Delta_D$ with eigenvalue λ.

Thus the homogeneous final value problem (1) has the above u as a *basic* solution if, coincidentally, the final data u_T equals the eigenfunction v. Our construction includes the set $\mathscr{B}$ of such basic solutions u, its linear hull $\mathscr{E} = \operatorname{span} \mathscr{B}$ and a certain completion $\overline{\mathscr{E}}$.

Using the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \dots$ and the associated $L_2(\Omega)$-orthonormal basis $e_1, e_2, \dots$ of eigenfunctions of $-\Delta_D$, the space $\mathscr{E}$ (that corresponds to data $u_T \in \operatorname{span}(e_j)$) clearly consists of solutions u being *finite* sums

$$u(t,x) = \textstyle\sum_j \mathrm{e}^{(T-t)\lambda_j} (u_T \mid e_j) e_j(x). \tag{2}$$

So at $t = 0$ there is, by the finiteness, a vector $u(0,x)$ in $L_2(\Omega)$ fulfilling

$$\|u(0,\cdot)\|^2 = \textstyle\sum_j \mathrm{e}^{2T\lambda_j} |(u_T \mid e_j)|^2 < \infty. \tag{3}$$

When summation is extended to all $j \in \mathbb{N}$, condition (3) becomes very strong, as it is only satisfied for special u_T: by Weyl's law $\lambda_j = \mathscr{O}(j^{2/n})$, so a single term in (3) yields $|(u_T \mid e_j)| \le c \exp(-T j^{2/n})$; whence the L_2-coordinates of such u_T decay rapidly for $j \to \infty$. This has been known since the 1950s; cf. the work of John [9] and Miranker [12].

More recently e.g. Isakov [7] emphasized the observation, made already in [12], that (2) gives rise to an *instability*: the sequence of data $u_{T,k} = e_k$ has length 1 for all k, but (2) gives $\|u_k(0,\cdot)\| = \|\mathrm{e}^{T\lambda_k} e_k\| = \mathrm{e}^{T\lambda_k} \nearrow \infty$ for $k \to \infty$. Thus (1) is not well-posed in $L_2(\Omega)$.

In general, this instability shows that the L_2-norm is an insensitive choice. To obtain well-adapted spaces for (1) with $f = 0$, $g = 0$, one could depart from (3). Indeed, along with the solution space $\mathscr{E}$, a norm on the final data $u_T \in \operatorname{span}(e_j)$ can be *defined* by (3); and $|||u_T||| = (\sum_{j=1}^\infty \mathrm{e}^{2T\lambda_j} |(u_T \mid e_j)|^2)^{1/2}$ can be used as norm on the u_T that correspond to solutions u in the above completion $\overline{\mathscr{E}}$. This would give the well-posedness of the homogeneous version of (1) with $u \in \overline{\mathscr{E}}$. (Cf. [3].)

But we have first of all replaced specific eigenvalue distributions by using sesqui-linear forms, cf. Lax–Milgram's lemma, which allowed us to cover general elliptic operators A.

Secondly the *fully* inhomogeneous problem (1) is covered. Here it does not suffice to choose the norm on the data (f, g, u_T) suitably (cf. $|||u_T|||$), for one has to *restrict* (f, g, u_T) to a subspace first by imposing certain *compatibility conditions*. These have long been known for parabolic problems, but they have a new form for final value problems.

2. The abstract final value problem

Our main analysis concerns a (possibly non-selfadjoint) Lax–Milgram operator A defined in H from a bounded V-elliptic sesquilinear form $a(\cdot,\cdot)$ in a Gelfand triple, i.e. densely injected Hilbert spaces $V \hookrightarrow H \hookrightarrow V^*$ with norms $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$.

In this set-up, we consider the following general final value problem: given data $f \in L_2(0,T;V^*)$, $u_T \in H$, determine the vector distributions $u \in \mathscr{D}'(0,T;V)$ fulfilling

$$\left.\begin{aligned} \partial_t u + Au &= f && \text{in } \mathscr{D}'(0,T;V^*), \\ u(T) &= u_T && \text{in } H. \end{aligned}\right\} \tag{4}$$

A wealth of parabolic Cauchy problems with homogeneous boundary conditions have been efficiently treated using such triples (H, V, a) and the $\mathscr{D}'(0,T;V^*)$ framework in (4); cf. works of Lions and Magenes [10], Tanabe [14], Temam [15], Amann [2]. Also recently, e.g., Almog, Grebenkov, Helffer, Henry studied variants of the complex Airy operator via such triples [1,5,4], and our results should at least extend to final value problems for those of their realisations that have non-empty spectrum.

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