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Partial differential equations

On the local existence for the Euler equations with free boundary for compressible and incompressible fluids

Sur l'existence locale de solutions des équations d'Euler pour les fluides compressibles et incompressibles, avec frontière libre

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A R T I C L E I N F O

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A B S T R A C T

We consider the free boundary compressible and incompressible Euler equations with surface tension. In both cases, we provide a priori estimates for the local existence with the initial velocity in H^3, with the H^3 condition on the density in the compressible case. An additional condition is required on the free boundary. Compared to the existing literature, both results lower the regularity of initial data for the Lagrangian Euler equation with surface tension.

R É S U M É

Nous considérons les équations d'Euler compressibles et incompressibles avec frontière libre et tension de surface. Dans les deux cas, nous fournissons des estimations *a priori* pour l'existence de solutions locales avec vitesse initiale dans H^3 et la condition H^3 sur la densité dans le cas compressible. Une condition supplémentaire est nécessaire sur la frontière libre. Par comparaison avec la littérature, les deux résultats abaissent la régularité des données initiales pour les équations d'Euler en coordonnées lagrangiennes, avec tension de surface.

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1. Introduction

In this note, we address the water wave problem, which has been studied extensively. Our setting is rather general: we consider the Euler equations with a free surface and allow initial data to have nonzero curl, i.e. the initial data is rotational. The domain is assumed to be of finite depth, but the results can be easily adapted to the infinite depth as well.

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We consider both the compressible and incompressible cases; for the compressible Euler equations, we assume that the density is bounded from below, i.e. we consider the case of a liquid.

Our aim in this note is to announce two recent results on the local existence with non-zero surface tension. In both main theorems (the compressible and incompressible cases, respectively), we assume that the initial data is of H^3 regularity in the interior. This lowers the regularity of existing results for the compressible equations. In the incompressible setting, our result lowers the known regularity in Lagrangian coordinates, albeit it does not improve over what has been obtained in Eulerian coordinates.

The history of the Euler equations with free interface is rich—we refer the reader to [2,4,5] for a more complete account. We only mention a few important works dealing with non-zero surface tension. The first general well-posedness result for the incompressible free-boundary Euler equations with surface tension is [2], followed by [9,10] and [3]. Earlier, well-posedness had been established under the assumption that the initial vorticity vanishes on the boundary [8]. For the compressible equations with surface tension, the first result was [1].

In the first part of the note, we address the compressible Euler equations, while in the second part we treat the incompressible version.

2. Compressible case

In the Lagrangian setting, the free-surface compressible Euler equations read

$$R\partial_t v^\alpha + a^{\mu\alpha}\partial_\mu q = 0 \qquad \text{in } [0,T)\times\Omega, \tag{1a}$$

$$\partial_t R + R a^{\mu\alpha}\partial_\mu v_\alpha = 0 \qquad \text{in } [0,T)\times\Omega, \tag{1b}$$

$$\partial_t a^{\alpha\beta} + a^{\alpha\gamma}\partial_\mu v_\gamma a^{\mu\beta} = 0 \qquad \text{in } [0,T)\times\Omega, \tag{1c}$$

$$q = q(R) \qquad \text{in } [0,T)\times\Omega, \tag{1d}$$

$$a^{\mu\alpha}N_\mu q + \sigma|a^T N|\Delta_g \eta^\alpha = 0 \qquad \text{on } [0,T)\times\Gamma_1, \tag{1e}$$

$$v^\mu N_\mu = 0 \qquad \text{on } [0,T)\times\Gamma_0, \tag{1f}$$

$$\eta(0,\cdot) = \text{id}, \quad R(0,\cdot) = \varrho_0, \quad v(0,\cdot) = v_0. \tag{1g}$$

Above, v, R, and q denote the Lagrangian velocity, density, and the pressure, respectively; N is the unit outer normal to $\partial\Omega$, a is the inverse of $\nabla\eta$, $\sigma = \text{constant} > 0$ is the coefficient of surface tension, and Δ_g is the Laplacian of the metric g_{ij} induced on $\partial\Omega(t)$ by the embedding η, i.e. $g_{ij} = \partial_i\eta\cdot\partial_j\eta = \partial_i\eta^\mu\partial_j\eta_\mu$.

We consider the domain $\Omega_0 \equiv \Omega = \mathbb{T}^2\times(0,1)$. We note that using the straightening map in [7, Remark 4.2], it is easy to modify the approach to consider a general curved domain $\Omega' = \mathbb{R}^2\times(0,h(x_1,x_2))$. Applying the change of variable in [7], we get

$$R\partial_t v^\alpha + a^{\mu\alpha}b^\beta{}_\mu\partial_\beta q = 0$$

instead of (1a); here b is the cofactor inverse matrix of the straightening map. The other equations in the system (1a)–(1g) are modified similarly. The methods outlined here then easily carry over for the new system as well, provided Ω' is at least H^5 regular.

Denoting the coordinates on Ω by (x^1,x^2,x^3), we have $\Gamma_1 = \mathbb{T}^2\times\{x^3=1\}$ as the free boundary and $\Gamma_0 = \mathbb{T}^2\times\{x^3=0\}$ as the stationary one. On the pressure function, we assume

$$\left(\frac{q(R)}{R}\right)' \geq A_q = \text{constant} > 0, \tag{2}$$

which is satisfied by a large class of equations of state.

We denote by Π the canonical projection, on $\eta(\Gamma_1)$, from the tangent bundle of $\eta(\Omega)$ to its normal bundle, which is given by $\Pi^\alpha_\beta = \delta^\alpha_\beta - g^{kl}\partial_k\eta^\alpha\partial_l\eta_\beta$. We recall that initial data for (2) is required to satisfy compatibility conditions (cf. [5]). The following is the main result in the compressible case.

Theorem 2.1. *Let v_0 be a smooth vector field on Ω and ϱ_0 a smooth positive function on Ω bounded away from zero from below. Assume that v_0 and ϱ_0 satisfy the compatibility conditions. Let $q\colon(0,\infty)\to(0,\infty)$ be a smooth function satisfying (2) in a neighborhood of ϱ_0. Then, there exist a $T_* > 0$ and a constant C_*, depending only on $\sigma > 0$, $\|v_0\|_3$, $\|v_0\|_{3,\Gamma_1}$, $\|\varrho_0\|_3$, $\|\varrho_0\|_{3,\Gamma_1}$, and $\|(\Delta\,\mathrm{div}\,v_0)\restriction\Gamma_1\|_{-1,\Gamma_1}$ such that any smooth solution (v,R) to (1), with initial condition (v_0,ϱ_0) and defined on the time interval $[0,T_*)$, satisfies*

$$\begin{aligned}\mathscr{N}(t) &= \|v\|_3^2 + \|\partial_t v\|_2^2 + \|\partial_t^2 v\|_1^2 + \|\partial_t^3 v\|_0^2 + \|R\|_3^2 + \|\partial_t R\|_2^2 \\ &\quad + \|\partial_t^2 R\|_1^2 + \|\partial_t^3 R\|_0^2 + \|\Pi\overline{\partial}\partial_t^2 v\|_{0,\Gamma_1}^2 + \|\Pi\overline{\partial}^2\partial_t v\|_{0,\Gamma_1}^2 \leq C_*,\end{aligned} \tag{3}$$

where $\overline{\partial}$ stands for the tangential derivative.