Doctopic: Number theory ARTICLE IN PRESS CRASS1:6043

C. R. Acad. Sci. Paris, Ser. I ••• (••••) •••–•••

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number theory

Multiplicative functions additive on generalized pentagonal numbers

Les fonctions multiplicatives qui sont additives sur les nombres pentagonaux généralisés

Byungchan Kim [a], Ji Young Kim [b], Chong Gyu Lee [c], Poo-Sung Park [d]

[a] *School of Liberal Arts, Seoul National University of Science and Technology, 232 Gongneung-ro, Nowon-gu, Seoul 01811, Republic of Korea*
[b] *Department of Mathematical Sciences, Seoul National University, 1 Gwanak-ro, Gwanak-Gu, Seoul 08826, Republic of Korea*
[c] *Department of Mathematics, Soongsil University, 369 Sangdo-ro, Dongjak-gu, Seoul 06978, Republic of Korea*
[d] *Department of Mathematics Education, Kyungnam University, Changwon-si, Gyeongsangnam-do 51767, Republic of Korea*

A R T I C L E I N F O

Article history:
Received 15 October 2017
Accepted after revision 22 December 2017
Available online xxxx

Presented by the Editorial Board

A B S T R A C T

We prove that the set GP of all nonzero generalized pentagonal numbers is an additive uniqueness set; if a multiplicative function f satisfies the equation

$$f(a+b) = f(a) + f(b),$$

for all $a, b \in GP$, then f is the identity function.

R É S U M É

Nous prouvons que l'ensemble GP de tous les nombres pentagonaux généralisés non nuls est un ensemble d'unicité additive ; si une fonction multiplicative f satisfait l'équation

$$f(a+b) = f(a) + f(b),$$

pour tous $a, b \in GP$, alors f est la fonction identité.

1. Introduction

An arithmetic function $f : \mathbb{Z}^+ \longrightarrow \mathbb{C}$ is called *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. In 1992, Spiro proved that if a multiplicative function f satisfies $f(p_0) \neq 0$ for some prime p_0 and

E-mail addresses: bkim4@seoultech.ac.kr (B. Kim), jykim98@snu.ac.kr, jykim.math@gmail.com (J.Y. Kim), cglee@ssu.ac.kr (C.G. Lee), pspark@kyungnam.ac.kr (P.-S. Park).

https://doi.org/10.1016/j.crma.2017.12.011
1631-073X/

Please cite this article in press as: B. Kim et al., Multiplicative functions additive on generalized pentagonal numbers, C. R. Acad. Sci. Paris, Ser. I (2018), https://doi.org/10.1016/j.crma.2017.12.011

$$f(p+q) = f(p) + f(q) \text{ for all primes } p \text{ and } q,$$

then f is the identity function [9]. More generally, Spiro asked which subset E of $\mathbb{Z}^+$ could determine an arithmetic function f uniquely in $\mathcal{S}$ under conditions

$$f(a+b) = f(a) + f(b) \text{ for all } a, b \in E,$$

where $\mathcal{S}$ is a set of arithmetic functions. Such a set E is called an *additive uniqueness set* for $\mathcal{S}$ following Spiro's theme.

After Spiro's work, this interesting subject has been studied and extended in many directions (see [1], [2], [3], [4], [5], [6], [7], and [8], for example). In particular, Chung and Phong [3] showed that the set of all triangular numbers is an additive uniqueness set for multiplicative functions, while Chung [2] showed that the set of square numbers is not an additive uniqueness set for multiplicative functions.

So it is natural to examine pentagonal numbers. The nonzero *generalized pentagonal numbers* are the integers obtained by the formula

$$P_n = \frac{n(3n-1)}{2},$$

with $n = \pm 1, \pm 2, \pm 3, \ldots$. Let GP be the set of nonzero generalized pentagonal numbers;

$$GP = \{1, 2, 5, 7, 12, 15, 22, 26, 35, \ldots\}.$$

In this article, we prove that the set GP is an additive uniqueness set for multiplicative functions.

Theorem 1.1. *If a multiplicative function $f : \mathbb{Z}^+ \longrightarrow \mathbb{C}$ satisfies*

$$f(a+b) = f(a) + f(b),$$

for arbitrary generalized pentagonal numbers a, b, then f is the identity function.

2. Proof of Theorem 1.1

We will prove the Theorem using induction on n. We assume that $f(k) = k$ for all $k < n$. Since f is multiplicative, it suffices to prove the case that n is a prime power. For notational convenience, we let $P_m^{\epsilon} = \frac{m(3m+\epsilon)}{2}$ where $\epsilon \in \{-1, +1\}$.

Proposition 2.1. *P_n^{ϵ} is a product of two coprime numbers.*

Proof. If n is even, then

$$P_n^{\epsilon} = \frac{n}{2} \cdot (3n+\epsilon),$$

where $\gcd(\frac{n}{2}, 3n+\epsilon) \le \gcd(n, 3n+\epsilon) = \gcd(n, \epsilon) = 1$.

When n is odd,

$$P_n^{\epsilon} = n \cdot \frac{3n+\epsilon}{2},$$

where $\gcd(n, \frac{3n+\epsilon}{2}) \le \gcd(n, 3n+\epsilon) = 1$. □

Lemma 2.2. *Let $p \neq 5$ be a prime and let $r \in \mathbb{Z}^+$. Then there are $a, b \in GP$ and $\lambda \in \mathbb{Z}^+$ such that*

$$\lambda p^r = a + b,$$

where $\gcd(\lambda, p) = 1$ with $\lambda < p^r$. Moreover, a and b are products of coprime numbers which are smaller than p^r. Furthermore, the same statement is true for $p = 5$ with $r > 1$.

Proof. We split the proof into four cases: $p = 2$, $p = 3$, $p = 5$, and $p \ge 7$.

Case $p = 2$**:** Since $2^r \equiv \epsilon \pmod 3$, we can let $2^r = 3m + \epsilon$ for a positive odd integer m. Then

$$P_m^{\epsilon} + P_m^{\epsilon} = \frac{m(3m+\epsilon)}{2} + \frac{m(3m+\epsilon)}{2} = m(3m+\epsilon) = m \cdot 2^r,$$

where the largest factor $\frac{3m+\epsilon}{2}$ of P_m^{ϵ} is smaller than 2^r. By letting $a = P_m^{\epsilon}$, $b = P_m^{\epsilon}$ and $\lambda = m = \frac{2^r - \epsilon}{3}$, we get a and b whose factors are smaller than 2^r and $\gcd(\lambda, 2) = 1$. Hence, the $p = 2$ case follows.

Please cite this article in press as: B. Kim et al., Multiplicative functions additive on generalized pentagonal numbers, C. R. Acad. Sci. Paris, Ser. I (2018), https://doi.org/10.1016/j.crma.2017.12.011