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Mathematical analysis

Non-uniformly hyperbolic horseshoes in the standard family

Fers à cheval hyperboliques non uniformes dans la famille standard

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A R T I C L E I N F O

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A B S T R A C T

We show that the non-uniformly hyperbolic horseshoes of Palis and Yoccoz occur in the standard family of area-preserving diffeomorphisms of the two-torus.

R É S U M É

Nous montrons que les fers à cheval hyperboliques non uniformes de Palis et Yoccoz apparaissent dans la famille standard des difféomorphismes du tore de dimension 2 préservant l'aire.

1. Introduction

In their *tour-de-force* work about the dynamics of surface diffeomorphisms, Palis and Yoccoz [2] proved that the so-called *non-uniformly hyperbolic horseshoes* are very frequent in the generic unfolding of a first heteroclinic tangency associated with periodic orbits in a horseshoe with Hausdorff dimension slightly bigger than one.

In the same article, Palis and Yoccoz gave an *ad hoc* example of a 1-parameter family of diffeomorphisms of the two-sphere fitting the setting of their main results, and thus exhibiting non-uniformly hyperbolic horseshoes: see page 3 (and, in particular, Figure 1) of [2].

In this note, we show that the *standard family* $f_k : \mathbb{T}^2 \to \mathbb{T}^2$, $k \in \mathbb{R}$,

$$f_k(x, y) := (-y + 2x + k\sin(2\pi x), x)$$

of area-preserving diffeomorphisms of the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ displays non-uniformly hyperbolic horseshoes.

More precisely, our main theorem is:

Theorem 1.1. *There exists $k_0 > 0$ such that, for all $|k| > k_0$, the subset of parameters $r \in \mathbb{R}$ such that $|r - k| < 4/k^{1/3}$ and f_r exhibits a non-uniformly hyperbolic horseshoe (in the sense of Palis–Yoccoz [2]) has positive Lebesgue measure.*

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The remainder of this text is divided into three sections: in Section 2, we briefly recall the context of Palis–Yoccoz work [2]; in Section 3, we revisit some elements of Duarte's construction [1] of tangencies associated with certain (uniformly hyperbolic) horseshoes of f_k; finally, we establish Theorem 1.1 in Section 4 by modifying Duarte's constructions (from Section 3) in order to apply the Palis–Yoccoz results (from Section 2).

2. Non-uniformly hyperbolic horseshoes

Suppose that F is a smooth diffeomorphism of a compact surface M displaying a first heteroclinic tangency associated with periodic points of a horseshoe K, that is:

- $p_s, p_u \in K$ belong to distinct periodic orbits of F;
- $W^s(p_s)$ and $W^u(p_u)$ have a quadratic tangency at a point $q \in M \setminus K$;
- for some neighborhoods U of K and V of the orbit $\mathcal{O}(q)$, the maximal invariant set of $U \cup V$ is precisely $K \cup \mathcal{O}(q)$.

Assume that K is *slightly thick* in the sense that its stable and unstable dimensions d^s and d^u satisfy $d_s + d_u > 1$ and

$$(d_s + d_u)^2 + \max(d_s, d_u)^2 < d_s + d_u + \max(d_s, d_u)$$

Remark 2.1. Since the stable and unstable dimensions of a horseshoe of an *area-preserving* diffeomorphism F always coincide, a slightly thick horseshoe K of an area-preserving diffeomorphism F has stable and unstable dimensions:

$$0.5 < d_s = d_u < 0.6$$

In this setting, the results proved by Palis and Yoccoz [2] imply the following statement.

Theorem 2.2 *(Palis–Yoccoz). Given a 1-parameter family $(F_t)_{|t|<t_0}$ with $F_0 = F$ and generically unfolding the heteroclinic tangency at q, the subset of parameters $t \in (-t_0, t_0)$ such that F_t has a non-uniformly hyperbolic horseshoe*[1] *has positive Lebesgue measure.*

3. Horseshoes and tangencies in the standard family

The standard family f_k generically unfolds tangencies associated with very thick horseshoes Λ_k: this phenomenon was studied in details by Duarte [1] during his proof of the almost denseness of elliptic islands of f_k for large generic parameters k.

In the sequel, we review some facts from Duarte's article about Λ_k and its tangencies (for later use in the proof of our Theorem 1.1).

For technical reasons, it is convenient to work with the standard family f_k and their *singular* perturbations

$$g_k(x, y) = (-y + 2x + k \sin(2\pi x) + \rho_k(x), x),$$

where ρ_k is defined in Section 4 of [1]. Here, it is worth to recall that the key features of ρ_k are:

- ρ_k has *poles* at the critical points $\nu_\pm = \pm 1/4 + O(1/k)$ of the function $2x + k \sin(2\pi x)$;
- ρ_k vanishes outside $|x \pm \frac{1}{4}| \leq \frac{2}{k^{1/3}}$.

In Section 2 of [1], Duarte constructs the stable and unstable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ for g_k. As it turns out, $\mathcal{F}^s$, resp. $\mathcal{F}^u$, is an almost vertical, resp. horizontal, foliation in the sense that it is generated by a vector field $(\alpha^s(x, y), 1)$, resp. $(1, \alpha^u(x, y))$, satisfying all properties described in Section 2 of Duarte's paper [1]. In particular, $\mathcal{F}^s$, resp. $\mathcal{F}^u$, describe the local stable, resp. unstable, manifolds for the standard map f_k at points whose future, resp. past, orbits stay in the region $\{f_k = g_k\}$, resp. $\{f_k^{-1} = g_k^{-1}\}$.

In Section 3 of [1], Duarte analyzes the projections π^s and π^u obtained by thinking the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ as fibrations over the singular circles $C_s = \{(x, \nu_+) \in \mathbb{T}^2\}$ and $C_u = \{(\nu_+, y) \in \mathbb{T}^2\}$. Among many things, Duarte shows that the circle map $\Psi_k : \mathbb{S}^1 \to \mathbb{S}^1$ defined by

$$(\Psi_k(x), \nu_+) := \pi^s(g_k(x, \nu_+)) \text{ or, equivalently, } (\nu_+, \Psi_k(y)) = \pi^u(g_k^{-1}(\nu_+, y))$$

is *singular* expansive with small distortion.

[1] We are not going to recall the definition of non-uniformly hyperbolic horseshoes here: instead, we refer the reader to the original article [2] for the details.

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