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Harmonic analysis

A note on weighted bounds for rough singular integrals

Une note sur les bornes pondérées pour les intégrales singulières rugueuses

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ABSTRACT

We show that the $L^2(w)$ operator norm of the composition $M \circ T_\Omega$, where M is the maximal operator and T_Ω is a rough homogeneous singular integral with angular part $\Omega \in L^\infty(S^{n-1})$, depends quadratically on $[w]_{A_2}$, and that this dependence is sharp.

RÉSUMÉ

Nous montrons que la norme d'opérateur $L^2(w)$ du composé $M \circ T_\Omega$, où M est l'opérateur maximal et T_Ω est une intégrale singulière homogène rugueuse de partie angulaire $\Omega \in L^\infty(S^{n-1})$, dépend de manière quadratique de $[w]_{A_2}$ et que cette dépendance est précise.

1. Introduction

Consider a class of rough homogeneous singular integrals defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) \frac{\Omega(y/|y|)}{|y|^n} \mathrm{d}y,$$

with $\Omega \in L^\infty(S^{n-1})$ and having zero average over the sphere.

In [7], Hytönen, Roncal and Tapiola proved that

$$\|T_\Omega\|_{L^2(w)\to L^2(w)} \le C_n \|\Omega\|_{L^\infty} [w]_{A_2}^2, \tag{1.1}$$

where $[w]_{A_2} = \sup_Q \frac{\int_Q w \int_Q w^{-1}}{|Q|^2}$. Different proofs of this result, via a sparse domination, were given by Conde-Alonso, Culiuc, Di Plinio, and Ou [3], and by the author [8]. Recently, (1.1) was extended to maximal singular integrals by Di Plinio, Hytönen, and Li [4].

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It was conjectured in [7] that the quadratic dependence on $[w]_{A_2}$ in (1.1) can be improved to the linear one. In this note, we obtain a strengthening of (1.1), which, to a certain extent, supports this conjecture.

Theorem 1.1. *For every $w \in A_2$, we have*

$$\|M \circ T_\Omega\|_{L^2(w)\to L^2(w)} \le C_n \|\Omega\|_{L^\infty}[w]_{A_2}^2, \tag{1.2}$$

and this bound is optimal, in general.

Here M denotes the standard Hardy–Littlewood maximal operator. Notice that $\|M\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}$, and this bound is sharp [1]. Therefore, (1.2) cannot be obtained via a simple combination of the sharp linear bound for M with (1.1). The proof of (1.2) is based essentially on the technique introduced in [8].

2. Preliminaries

Recall that a family of cubes $\mathcal{S}$ is called sparse if there exists $0 < \alpha < 1$ such that for every $Q \in \mathcal{S}$, one can find a measurable set $E_Q \subset Q$ with $|E_Q| \ge \alpha|Q|$, and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint.

Given a sublinear operator T, define the maximal operator $M_{p,T}$ by

$$M_{p,T} f(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |T(f\chi_{\mathbb{R}^n\setminus 3Q})|^p \mathrm{d}y\right)^{1/p}.$$

Denote $\langle f\rangle_{p,Q} = \left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p}$.

Proposition 2.1. *Assume that T and $M_{p,T}$ are of weak type $(1,1)$ and, moreover, $\|M_{p,T}\|_{L^1\to L^{1,\infty}} \le Kp$ for all $p \ge 2$. Then*

$$\|T\|_{L^2(w)\to L^2(w)} \le C_n(\|T\|_{L^1\to L^{1,\infty}} + K)[w]_{A_2}^2. \tag{2.1}$$

Proof. This is just a combination of several known facts. By [8, Cor. 3.2], for every suitable f, g, there exists a sparse family $\mathcal{S}$ such that

$$|\langle Tf, g\rangle| \le C_n(\|T\|_{L^1\to L^{1,\infty}} + Kp')\sum_{Q\in\mathcal{S}}\langle f\rangle_{1,Q}\langle g\rangle_{p,Q}|Q| \quad (p>1).$$

But it was shown in [3] (see the proof of Corollary A1 there) that this sparse bound implies (2.1). □

In particular, T_Ω with $\Omega \in L^\infty$ satisfies the hypothesis of Proposition 2.1, namely, it was proved in [8] that

$$\|M_{p,T_\Omega} f\|_{L^{1,\infty}} \le C_n\|\Omega\|_{L^\infty} p\|f\|_{L^1} \quad (p \ge 1). \tag{2.2}$$

3. Proof of Theorem 1.1

First, by a general extrapolation argument found in [9], the sharpness of (1.2) follows from $\|M\circ T_\Omega\|_{L^p\to L^p} \ge \frac{c}{(p-1)^2}$ as $p \to 1$. The latter relation holds for a subclass of T_Ω with kernels satisfying the standard nondegeneracy assumptions. In particular, it can be easily checked for the Hilbert transform.

Turn to the proof of (1.2). By homogeneity, one can assume that $\|\Omega\|_{L^\infty} = 1$. The proof is based on two pointwise estimates:

$$M(T_\Omega f)(x) \lesssim MMf(x) + M_{1,T_\Omega} f(x) \tag{3.1}$$

and

$$M_{p,(M_{1,T_\Omega})} f(x) \lesssim Mf(x) + M_{p,T_\Omega} f(x) \quad (p \ge 2) \tag{3.2}$$

(we use the usual notation $A \lesssim B$ if $A \le C_n B$).

Let us show first how to complete the proof using these estimates. By (2.2), M_{1,T_Ω} is of weak type $(1,1)$. Applying (2.2) again along with (3.2) yields $\|M_{p,(M_{1,T_\Omega})}\|_{L^1\to L^{1,\infty}} \lesssim p$. Therefore, by Proposition 2.1,

$$\|M_{1,T_\Omega}\|_{L^2(w)\to L^2(w)} \lesssim [w]_{A_2}^2.$$

This estimate, combined with (3.1) and Buckley's linear bound for M [1], implies (1.2).