C. R. Acad. Sci. Paris, Ser. I 356 (2018) 227–234

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Mathematical problems in mechanics/Differential geometry

A nonlinear shell model of Koiter's type

Un modèle non linéaire de coques de type Koiter

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ARTICLE INFO

Article history:
Received 14 December 2017
Accepted after revision 14 December 2017
Available online 5 January 2018

Presented by Philippe G. Ciarlet

ABSTRACT

We define a new two-dimensional nonlinear shell model "of Koiter's type" that can be used for the modeling of any type of shell and boundary conditions and for which we establish an existence theorem. The model uses a specific three-dimensional stored energy function of Ogden's type that satisfies all the assumptions of John Ball's fundamental existence theorem in three-dimensional nonlinear elasticity and that is adapted here to the modeling of thin nonlinearly elastic shells by means of specific deformations that are quadratic with respect to the transverse variable.

RÉSUMÉ

Nous définissons un nouveau modèle bidimensionnel non linéaire de coques « de type Koiter » qui peut être utilisé pour la modélisation de tout type de coque et de conditions aux limites et pour lequel nous établissons un théorème d'existence. Ce modèle utilise une densité d'énergie de type Ogden satisfaisant toutes les hypothèses du théorème d'existence fondamental de John Ball en élasticité tridimensionnelle non linéaire et qui est adaptée ici à la modélisation des coques non linéairement élastiques minces au moyen de déformations particulières, qui sont quadratiques en la variable transverse.

1. Preliminaries

Greek indices and exponents vary in the set $\{1,2\}$, Latin indices and exponents vary in the set $\{1,2,3\}$, and the summation convention for repeated indices and exponents is used in conjunction with these rules. Boldface letters are used to designate vector and matrix fields.

The three-dimensional Euclidean space is denoted $\mathbb{E}^3$. The inner product, the exterior product, and the norm, in $\mathbb{E}^3$ are respectively denoted $\cdot$, $\wedge$, and $|\cdot|$. The space of real $n \times n$ matrices is denoted $\mathbb{M}^n$ and the Frobenius norm in $\mathbb{M}^n$ is denoted $\|\cdot\|$. A matrix in $\mathbb{M}^n$ with components g_{ij} is denoted (g_{ij}), the first index (here i) indicating the row in the matrix.

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https://doi.org/10.1016/j.crma.2017.12.005

A *domain* in $\mathbb{R}^2$ is a bounded and connected open subset of $\mathbb{R}^2$ whose boundary is Lipschitz-continuous in the sense of Nečas [15].

Given an open subset ω of $\mathbb{R}^2$, a finite dimensional real space $\mathbb{Y}$, any $p \geq 1$ and any integer $m \geq 0$, the notation $\mathcal{C}^m(\overline{\omega}; \mathbb{Y})$, resp. $W^{m,p}(\omega; \mathbb{Y})$, denotes the space of $\mathbb{Y}$-valued fields with components in $\mathcal{C}^m(\overline{\omega})$, resp. in the Sobolev space $W^{m,p}(\omega)$.

Given an open subset ω of $\mathbb{R}^2$, we let $y = (y_\alpha)$ denote a generic point in ω and we let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$. An *immersion* from $\overline{\omega}$ into $\mathbb{E}^3$ is a mapping $\boldsymbol{\psi} \in \mathcal{C}^1(\overline{\omega}; \mathbb{E}^3)$ such that the two vector fields $\partial_\alpha \boldsymbol{\psi} : \overline{\omega} \to \mathbb{E}^3$ are linearly independent at each point of $\overline{\omega}$. The image $\boldsymbol{\psi}(\overline{\omega})$ of $\overline{\omega}$ by $\boldsymbol{\psi}$ is a *surface* (with boundary) in $\mathbb{E}^3$.

Given an immersion $\boldsymbol{\psi} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$, we let

$$\boldsymbol{a}_3(\boldsymbol{\psi}) := \frac{\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}}{|\partial_1 \boldsymbol{\psi} \wedge \partial_2 \boldsymbol{\psi}|}, \; a_{\alpha\beta}(\boldsymbol{\psi}) := \partial_\alpha \boldsymbol{\psi} \cdot \partial_\beta \boldsymbol{\psi}, \; b_{\alpha\beta}(\boldsymbol{\psi}) := \partial_{\alpha\beta} \boldsymbol{\psi} \cdot \boldsymbol{a}_3(\boldsymbol{\psi}), \text{ and } a(\boldsymbol{\psi}) := \det(a_{\alpha\beta}(\boldsymbol{\psi}));$$

the functions $a_{\alpha\beta}(\boldsymbol{\psi})$ and $b_{\alpha\beta}(\boldsymbol{\psi})$ respectively denote the covariant components of the *first fundamental form* and those of the *second fundamental form* along the surface $\boldsymbol{\psi}(\overline{\omega})$.

Given an immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ considered as fixed, we let (for brevity)

$$\boldsymbol{a}_3 := \boldsymbol{a}_3(\boldsymbol{\theta}), \quad a_{\alpha\beta} := a_{\alpha\beta}(\boldsymbol{\theta}), \quad b_{\alpha\beta} := b_{\alpha\beta}(\boldsymbol{\theta}), \text{ and } a := a(\boldsymbol{\theta}),$$

and, given any arbitrary immersion $\boldsymbol{\psi} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$, we let

$$G_{\alpha\beta}(\boldsymbol{\psi}) := \frac{1}{2}(a_{\alpha\beta}(\boldsymbol{\psi}) - a_{\alpha\beta}) \text{ and } R_{\alpha\beta}(\boldsymbol{\psi}) := b_{\alpha\beta}(\boldsymbol{\psi}) - b_{\alpha\beta}$$

respectively denote the covariant components of the *change of metric tensor* and those of the *change of curvature tensor* between the surfaces $\boldsymbol{\theta}(\overline{\omega})$ and $\boldsymbol{\psi}(\overline{\omega})$.

Given a domain $\omega \subset \mathbb{R}^2$, a "small" parameter $\varepsilon > 0$, and an immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$, we let

$$\Omega := \omega \times]-\varepsilon, \varepsilon[\text{ and } \boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{E}^3)$$

denote the extension of the mapping $\boldsymbol{\theta}$ defined by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ at each } y \in \overline{\omega} \text{ and each } x_3 \in [-\varepsilon, \varepsilon].$$

Then one can show that, if $\varepsilon > 0$ is small enough, $\det \boldsymbol{\nabla\Theta} > 0$ in $\overline{\Omega}$ and $\boldsymbol{\Theta}$ is injective over $\overline{\Omega}$; cf. Theorem 4.1-1 in [5].

We let $x = (x_i)$ with $(x_\alpha) = (y_\alpha) \in \overline{\omega}$ and $x_3 \in [-\varepsilon, \varepsilon]$ denote a generic point in $\overline{\Omega}$, we let $\partial_i := \partial/\partial x_i$, and we let

$$g_{ij} := \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta} \text{ and } (g^{kl}) := (g_{ij})^{-1}$$

respectively denote the covariant and contravariant components of the *metric tensor field* associated with the mapping $\boldsymbol{\Theta}$.

Finally, we let

$$A^{ijkl} := \lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{jk}) \text{ in } \overline{\Omega},$$

and

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}) \text{ in } \overline{\omega},$$

respectively denote the contravariant components of the *three-dimensional*, and *two-dimensional*, *elasticity tensor* associated with an elastic material with *Lamé constants* λ and μ. If $3\lambda + 2\mu > 0$ and $\mu > 0$, both tensors are uniformly positive-definite, in the sense that there exist two constants $C_0 > 0$ and $c_0 > 0$ depending on λ and μ such that

$$C_0 \sum_{i,j} (t_{ij})^2 \leq A^{ijkl}(x) t_{kl} t_{ij} \text{ for all } x \in \overline{\Omega} \text{ and all symmetric } 3 \times 3 \text{ tensors } (t_{ij}),$$

and

$$c_0 \sum_{\alpha,\beta} (s_{\alpha\beta})^2 \leq a^{\alpha\beta\sigma\tau}(y) s_{\sigma\tau} s_{\alpha\beta} \text{ for all } y \in \overline{\omega} \text{ and all symmetric } 2 \times 2 \text{ tensors } (s_{\alpha\beta});$$

(cf. Theorems 3.9-1 and 4.4-1 in [5]). Note, however, that the Lamé constants of all known elastic materials satisfy the stronger assumptions $\lambda > 0$ and $\mu > 0$.