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Bessel sequences, wavelet frames, duals and extensions

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Abstract

From the perspectives of duality and extensions, Gabor frames and wavelet frames have contrasting behaviour. Our chief concern here is about duality. Canonical duals of wavelet frames may not be wavelet frames, whereas canonical duals of Gabor frames are Gabor frames. Keeping these in view, we give several constructions of wavelet frames with wavelet canonical duals. For this, a simple characterisation of Bessel sequences and a general commutativity result are given, the former also leading naturally to some extension results.

Keywords: Bessel sequence; Frame; Canonical dual; Frame operator; Gabor frame; Wavelet frame

1. Introduction

Time–frequency analysis studies two types of coherent systems, the Gabor and the wavelet systems. The two have some similarities — for example both are generated by a single function, Gabor frames through modulations and translations and wavelet frames through dilations and translations. These systems have some dissimilarities too. For instance, the former is well behaved *vis à vis* the canonical dual and 'extension property'. The canonical dual of a Gabor frame is again a Gabor frame, whereas the class of wavelet frames is not closed under taking canonical duals, as Daubechies ([6,7]) has shown (see also [1,3,11] for a discussion of her

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example). It is not difficult to show that any two Gabor Bessel sequences with the same time–frequency parameters have extensions to a pair of dual Gabor frames, but the corresponding extension problem is open for wavelet frames, see [2].

This paper has two basic purposes: to have a fresh look at extensions from a different perspective and to seek wavelet frames whose canonical duals have the same wavelet structure (*'wavelet frames with wavelet canonical duals'*). We obtain various extension results as natural consequences of a simple characterisation of Bessel sequences and 'mixed Bessel sequence operators'. Towards the second aim, we succeed in constructing several classes of wavelet frames with wavelet canonical duals.

After a section on preliminaries following this introduction, we present a characterisation of Bessel sequences in terms of orthonormal bases. Some known results on extensions of Bessel sequences are simple consequences.

Commutativity properties of operators are important in frame theory. Gabor frame operators have a certain nice commutativity property whereas such a property fails for wavelet frame operators. This is the underlying reason why the class of Gabor frames is closed under canonical duals, while this is not so for the class of wavelet frames. So, we take a closer look at certain commutativity properties and present unified, general commutativity results that subsume such results for Gabor and wavelet frames. These considerations lead us to constructions of wavelet frames with wavelet canonical duals in the last part of the paper.

2. Preliminaries

Frames are generalisations of orthonormal bases in Hilbert spaces. A *frame* in a (necessarily separable) Hilbert space $\mathcal{H}$ is a countable family $\{u_k\}$ satisfying two inequalities – a lower and an upper – of the form

$$\alpha\|f\|^2 \leq \Sigma|\langle f, u_k\rangle|^2 \leq \beta\|f\|^2$$

for some constants $0 < \alpha \leq \beta$ and for every $f \in \mathcal{H}$. When only the second of these inequalities is assumed, $\{u_k\}$ is called a *Bessel sequence*, so it is a kind of *semiframe*. For a Bessel sequence $\{u_k\}$, the series $\Sigma\langle f, u_k\rangle u_k$ converges unconditionally in $\mathcal{H}$ and if Sf denotes the sum, we obtain a bounded linear, positive operator S on $\mathcal{H}$. It may be called the *semiframe operator* of the Bessel sequence; the term 'Bessel sequence operator' is uncouth and the term 'frame operator' seems inappropriate unless $\{u_k\}$ is a frame. When $\{u_k\}$ is a frame, the operator S is also invertible and is called the *frame operator*. The image of a Bessel sequence under a bounded linear operator is also a Bessel sequence and the image of a frame under an invertible operator is a frame.

If $\{u_k\}$ is a frame with frame operator S, then the frame $\{S^{-1}u_k\}$ is called the *canonical dual frame* of $\{u_k\}$ in $\mathcal{H}$. A frame $\{u_k\}$ and its canonical dual together yield two different reconstruction formulas

$$f = \Sigma\langle f, S^{-1}u_k\rangle u_k = \Sigma\langle f, u_k\rangle S^{-1}u_k$$

for every $f \in \mathcal{H}$. In general, a frame $\{v_k\}$ is called a *dual* of a frame $\{u_k\}$ if $f = \Sigma\langle f, u_k\rangle v_k$ for all $f \in \mathcal{H}$. In this case, $\{u_k\}$ is also a dual of $\{v_k\}$ and we just say that they form a dual pair of frames (see Lemma 5.7.1 of [1]).

The space $L^2(\mathbb{R})$ admits some special kinds of frames of great importance in time–frequency analysis. These frames are got by applying, to a single function in $L^2(\mathbb{R})$, certain special classes of unitary operators, namely, *translations* T_α, *modulations* E_α and *dilations* D_α. These operators are defined, respectively, as follows. For $\alpha, t \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$,

$$T_\alpha f(t) = f(t-\alpha),\ \ E_\alpha f(t) = e^{2\pi i\alpha t} f(t) \text{ and } D_\alpha f(t) = \left(\frac{1}{\sqrt{|\alpha|}}\right) f(t/\alpha),\ \ (\alpha \neq 0).$$