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Virtual Special Issue - L.E.J. Brouwer after 50 years

On the natural concept of dimension

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Based on the invariance of the dimension number[2] one can define the dimension number of a manifold[3] as the number of parameters, by which a neighborhood of an arbitrary point can be represented injectively and continuously. However, according to *Poincaré*[4] this "arithmetic" definition takes our intuitive concept of space insufficiently into account. Poincaré therefore demands a recursive definition of approximately the following form[5]:

"A continuum is called n-dimensional, if one can divide it by one or more $(n-1)$*-dimensional continua into separate parts."*

Although the n-dimensional Jordan Theorem[6] indicates that such a definition is possible, it does not hold in the cited form.

First of all we remark that the notion of *continuum* should not be understood as an equivalent of *manifold*; in this case, the definition will only be useful once a characterization of manifolds as abstract sets is achieved, that is independent of the representation by parameters. Since this is presently not the case, in Poincaré's definition we must first of all propose a general description of continuum, such as this one: *"A normal set* (in the sense of *Fréchet*) π *is called a continuum, if*

[1] Note by the editors: This paper was first published in German in the Journal für die reine und angewandte Mathematik, 142 (1913), 146–152. It contains the first formal definition of dimension. We are indebted to the editors of the Journal für die reine und angewandte Mathematik for allowing us to reproduce it here and to Robbert Fokkink (TU Delft) for translating it into English. The present inclusion of this paper marks the beginning of the virtual special issue of Indagationes Mathematicae containing the papers collected under the header *L.E.J. Brouwer, fifty years later*, including those presented at the conference with the same name organized under auspices of the Koninklijk Wiskundig Genootschap on December 9, 2016, in Amsterdam.

[2] Compare my proof in Math. Annalen 70, p. 161–165 and its further development by *Lebesgue* in C.R. de l'Acad. des sciences, Paris, 27 mars 1911.

[3] For the definition of a manifold, see Math. Annalen 71, p. 97.

[4] Revue de métaphysique et de morale, 1912, p. 486, 487.

[5] ibid, p. 488.

[6] Compare the proof that is partly due to *Lebesgue* and partly due to myself in C.R. de l'Acad. des sciences, Paris, 27 mars 1911, and Math. Annalen 71, p. 305–319.

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for every pair of elements m_1 and m_2 there exists a connected, closed[7] *set, which is a subset of π and contains m_1 and m_2".*[8] For such general continua, which are not manifolds, our definition would lead to difficulties. For example, the cone of a Cartesian space would only be assigned dimension *one*, as it can be separated by a single point.

Also, the words "*one or more*" have to be altered, since more m-dimensional manifolds together can build an $m + p$-manifold.

All these defects can be remedied, if we alter Poincaré's definition as follows:

Let π be any normal set,[9] and let π_1, ρ, and ρ' be three subsets of π, which are closed[10] in π and are mutually disjoint. Then ρ and ρ' are called *separated in π by π_1* if every connected, closed subset of π which intersects both ρ and ρ' also intersects π_1. The expression: *"π has the general Dimensionsgrad n"*, in which n is an arbitrary natural number, should now say that for every choice of ρ and ρ' there exists a separating set π_1, which has the general Dimensionsgrad $n-1$, and that there exist ρ and ρ' which cannot be separated by a set of Dimensionsgrad less than $n - 1$. Furthermore, the expression: *"π has general Dimensionsgrad zero, or respectively, has infinite general Dimensionsgrad"* should mean that π does not contain a proper subcontinuum, or respectively, that π has neither general Dimensionsgrad zero nor has it general Dimensionsgrad equal to any natural number.[11]

It is easy to restate this definition in a non-recursive way. To this end, we imagine that the space π undergoes *dimension operations* by two players A and B, as follows: A selects two disjoint closed subsets ρ and ρ' in π, upon which B separates ρ and ρ' in π by a closed subset π_1. Then A selects two disjoint closed subsets ρ_1, ρ_1' in π_1, upon which B separates ρ_1 and ρ_2 in π_1 by a closed subset π_2. This process is repeated unrestrictedly, until eventually a subset π_k appears that no longer contains a proper subcontinuum. If B can make sure on the one hand that a set π_h appears with $h \leq n$, regardless of the choices of the ρ_ν and ρ_ν', and A can make sure on the other hand that no π_h appears such that $h < n$, regardless of the choice of π_ν, then we say that π has general Dimensionsgrad n. However, when there does not exist a natural number n such that B can make sure that π_h appears such that $h \leq n$, regardless of the choices of ρ_ν and ρ_ν', then we say that π *has infinite Dimensionsgrad*.

When a point P in π has neighborhoods that have general Dimensionsgrad m, but has no neighborhoods of a smaller general Dimensionsgrad, then we say that π *has Dimensionsgrad m in P*. The set may have different Dimensionsgrad in different points, but none of these can exceed the general Dimensionsgrad of the set. If the Dimensionsgrad is equal for all points in the set, then we say that it has a *homogeneous Dimensionsgrad.*

Based on the definitions above, Poincaré's requirement should now be fully substantiated by the following

Dimension Theorem. An n-dimensional manifold has homogeneous Dimensionsgrad n.[12]

To prove this theorem we first show that B can always make sure that $h \leq n$ in the dimension operations. To this end, B constructs a specific simplicial decomposition[13] ζ of π, once A has

[7] By closed set we mean here a set that contains its boundary elements, in which every infinite sequence of elements exhibits at least one boundary point.

[8] This definition is modeled on Schoenflies' definition of continua in n-dimensional space (cf. Bericht über die Lehre von Punktmannigfaltigkeiten, vol II, p. 117).

[9] In how far the definition applies to more general sets, should remain undiscussed here.

[10] π_1, ρ, and ρ' contain all their boundary points which are in π.

[11] According to this definition, both the Hilbert space and the Fréchet space R_ω have infinite general Dimensionsgrad.

[12] Since Dimensionsgrad is obviously an invariant of the Analysis Situs, the dimension theorem implies the invariance of the dimension number.

[13] Math. Annalen 71, p. 101.

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