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Stochastic processes of limited frequency and the effects of oversampling

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ABSTRACT

Discrete-time ARMA processes can be placed in a one-to-one correspondence with a set of continuous-time processes that are bounded in frequency by the Nyquist value of π radians per sample period. It is well known that, if data are sampled from a continuous process of which the maximum frequency exceeds the Nyquist value, then there will be a problem of aliasing. However, if the sampling is too rapid, then other problems will arise that may cause the ARMA estimates to be severely biased. The paper reveals the nature of these problems and it shows how they may be overcome.

1. Introduction

Modern digital communications depend on data sampled at regular intervals from continuously varying signals. The so-called sampling theorem defines the circumstances in which the continuous trajectory of the signal can be recovered from the discrete data. Although this crucial theorem is commonly attributed to Nyquist (1924, 1928) and to Shannon (1949), there are others who can reasonably lay claim to its discovery, as Luke (1999) has observed.

The sampling theorem indicates that, if at least two observations are made in the time that it takes the signal component of highest frequency to complete a single cycle, then a continuous signal can be reconstructed perfectly from the sampled sequence. In effect, the theorem poses a limit to the frequencies that can be captured by the sampled data. The limit is the so-called Nyquist frequency of π radians per sampling interval.

Sampling at an insufficient rate, also described as undersampling, leads to a problem of aliasing whereby elements of the signal of frequencies in excess of the Nyquist limit are confounded with elements of frequencies that lie within the observable range. A good account of how this arises has been provided by Oppenheim et al. (1983, Ch 8.), albeit that there are many other accessible sources.

The problem of aliasing is sometimes present in moving cinema pictures, which are created from the rapid projection of a succession of images that capture instants in the trajectories of moving objects. The problem is familiar to a generation who watched Western movies and who noticed the seemingly slow and sometimes retrograde motion in the depiction of the rapidly rotating wheels of a fleeing stagecoach.

The sampled data can be used to construct models of the processes generating the signals. Often, it is sufficient to model the relationships subsisting in the sampled data, with only a passing reference to the underlying continuous process. This may be achieved by fitting an autoregressive moving-average (ARMA) model to the data. However, dynamic systems may

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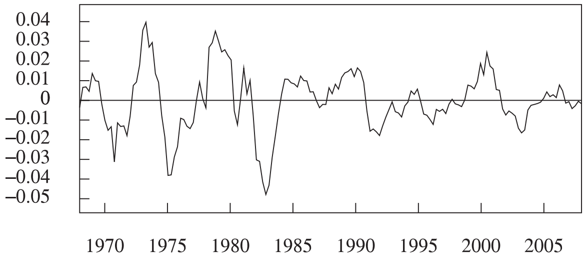

Fig. 1. The deviations of the logarithmic quarterly index of real US GDP from an interpolated trend. The observations are from 1968 to 2007. The trend is determined by a Hodrick–Prescott (Leser) filter with a smoothing parameter of 1600.

be described better by differential equations operating in continuous time. Therefore, the issue arises of how to make the translation from discrete-time data to a continuous-time model.

The usual methods for constructing continuous-time models from the discrete data are fraught with difficulties, and they depend on assumptions that may conflict with the evident properties of the signals. (A recent survey of these methods has been provided by Garnier et al., 2008.)

The usual assumption that underlies the estimation of stochastic differential equations is that their *primum mobile* or forcing function is a white-noise process consisting of a continuous stream of infinitesimal increments of a stochastic Wiener process. Such a white-noise process, which is everywhere continuous but nowhere differentiable, is unbounded in frequency. Thus, an assumption is commonly adopted that implies the inevitability of aliasing and that seems to preclude the possibility of estimating a model that faithfully represents the continuous process. However, the problems are not always so severe.

It can be shown that, if the poles of the transfer function that maps from the white-noise forcing function to the signal have frequency arguments that fall within the Nyquist range, then the stochastic differential equation can be estimated consistently. (See, for example, Pandit and Wu, 1983, Ch. 7) Also, it may be observed that if the transfer function of the model strongly attenuates the higher frequencies, then the assumption that the frequencies of the forcing function are unbounded is of limited significance.

Notwithstanding these easements in the estimation of stochastic differential equations, it is sometimes important to recognise the frequency limits of a stochastic forcing function. It is the purpose of this paper to highlight the problems that can arise when the frequency limit of the forcing function, and therefore of the signal, is less than the Nyquist value. Problems can then arise both with the discrete-time autoregressive moving-average model and with the corresponding continuous-time stochastic differential equation.

In some circumstances, it should be possible to reduce the rate of sampling so that the Nyquist frequency no longer exceeds the maximum frequency of the signal. Even when is it is not possible to vary the sampling rate directly, it may be possible to reconstitute the continuous signal in the manner indicated by the sampling theorem.

Then, it is possible to generate the sample points that would arise from resampling the continuous signal at an arbitrary rate. Thus, a sample can be obtained that is attuned to the maximum frequency within the signal. The relevant procedure will be described in this paper. However, a example will be presented for which it is sufficient to subsample the data by taking one in every four points.

The primary purpose of this paper is to illustrate the problems of over-rapid sampling, also described as oversampling, and to show how to overcome them. Nevertheless, it will be necessarily to deal more generally with the theory of autoregressive moving-average models and of the corresponding stochastic differential equations.

The plan of the paper is as follows. In Section 2, the problems of oversampling are illustrated in the context of a macroeconomic data sequence. In Section 3, an account is given of the essential sampling theorem; and it is shown how it can be adapted to finite data sequences. Section 4 provides the theory of frequency-limited continuous-time processes that bear a one-to-one correspondence with discrete-time ARMA models, and Section 5 explains the empirical findings of Section 2.

With these results in hand, it will be show, in Section 6, how it is possible, by means of a process of re-sampling, to obtain appropriate estimates of frequency-limited ARMA processes by means of the usual estimators of discrete-time ARMA models. Thereafter, it remains to describe, in Section 7, how a linear stochastic differential equation (LSDE) may be fitted to the frequency-limited data.

2. The effects of oversampling

The typical frequency-limited spectral structure can be illustrated by a data sequence that requires only a simple method of detrending. Fig. 1 shows the deviations from an interpolated trend of the logarithms of U.S. quarterly gross domestic product (GDP) for the period 1968–2007. The trend has been calculated using the filter of Hodrick and Prescott (1980, 1997) with the smoothing parameter set to the value of 1600. (The filter in question is also attributable to Leser, 1961.)

Fig. 2 shows the periodogram of the deviations. It will be seen that the essential spectral structure extends no further than the frequency of $\pi/4$ radians per quarter. The remainder of the periodogram comprises what may be described as

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