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Note

The diameter of strong orientations of Cartesian products of graphs

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ABSTRACT

Let G and H be graphs, and $G\Box H$ the Cartesian product of G and H. We prove that for every connected bridgeless graphs G and H, the Cartesian product $G\Box H$ admits an orientation of diameter at most $\mathrm{wdiam}_{\min}(G) + \mathrm{wdiam}_{\min}(H) + 8$, where $\mathrm{wdiam}_{\min}(G)$ denotes the minimum weak diameter of an orientation of G. Orientations of products of graphs that have bridges are considered as well, and an upper bound for the minimum diameter of such orientations is given.

1. Introduction

Let $D = (V, A)$ be a directed graph, and $u, v \in V$. If $(u, v) \in A$ we write $u \to v$, and we say that u is an *in-neighbor* of v, and that v is an *out-neighbor* of u. A *uv-path* in D is a sequence of pairwise distinct vertices $u = u_0, u_1, \ldots, u_n = v$ such that $u_iu_{i+1} \in A$ for all subscripts i. We say that D is *strong* if there is a uv-path in D for every $u, v \in V$. The *length* of the path $u = u_0, u_1, \ldots, u_n = v$ is n, the number of arcs between consecutive vertices. For vertices $u, v \in V$ the *distance* from u to v in D is the length of a shortest uv-path in D, if such a path exists, otherwise the distance is ∞. We denote the distance from u to v by $\mathrm{dist}_D(u, v)$, or simply by $\mathrm{dist}(u, v)$ when D is clear from the context. The *diameter* of D is

$$\mathrm{diam}(D) = \max\{\mathrm{dist}(u, v) \mid u, v \in V\}\,,$$

and the *weak diameter* of D is

$$\mathrm{wdiam}(D) = \max\{\min\{\mathrm{dist}(u, v), \mathrm{dist}(v, u)\} \mid u, v \in V\}\,.$$

Observe that for every pair of distinct vertices $u, v \in V$, there is a uv-path *and* a vu-path in D of length at most $\mathrm{diam}(D)$. Moreover, for every pair of distinct vertices $u, v \in V$, there is a uv-path *or* a vu-path in D of length at most $\mathrm{wdiam}(D)$. The difference between $\mathrm{diam}(D)$ and $\mathrm{wdiam}(D)$ can be arbitrarily large. To see this let D be a tournament with vertices $x_0, \ldots, x_n$, where (x_i, x_j) is an arc in D if and only if $j = i + 1$ or $j \le i - 2$. Clearly, $\mathrm{diam}(D) = n$ and $\mathrm{wdiam}(D) = 1$.

Let G be an undirected graph. Let $\mathrm{diam}_{\min}(G)$ be the minimum diameter of a strong orientation of G

$$\mathrm{diam}_{\min}(G) = \min\{\mathrm{diam}(D) \mid D \text{ is a strong orientation of } G\}\,.$$

Similarly let $\mathrm{wdiam}_{\min}(G)$ be the minimum weak diameter of a strong orientation of G

$$\mathrm{wdiam}_{\min}(G) = \min\{\mathrm{wdiam}(D) \mid D \text{ is a strong orientation of } G\}\,.$$

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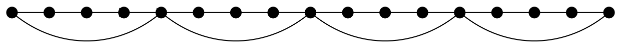

Fig. 1. The graph G_4.

For example, if C_n is the cycle on n vertices, then

$$\mathrm{diam}_{\min}(C_n) = n-1 \text{ and } \mathrm{wdiam}_{\min}(C_n) = \mathrm{diam}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor .$$

In the above example $\mathrm{wdiam}_{\min}$ is roughly one half of $\mathrm{diam}_{\min}$. We claim even more: there is a sequence of graphs G_n, such that

$$\lim_{n\to\infty} \frac{\mathrm{wdiam}_{\min}(G_n)}{\mathrm{diam}_{\min}(G_n)} = 0 .$$

To see this, let G_n be the graph obtained from a path on n^2+1 vertices $x_0, \ldots, x_{n^2}$, by adding edges $x_{n\ell}x_{n(\ell+1)}$ for $\ell = 0, \ldots, n-1$ (see Fig. 1). For any strong orientation of G_n the following is true: for every $\ell = 0, \ldots, n-1$ we have $x_{n\ell} \to x_{n(\ell+1)}$ if and only if $x_i \to x_{i-1}$ for $i = n\ell+1, \ldots, n(\ell+1)$, and $x_{n(\ell+1)} \to x_{n\ell}$ if and only if $x_{i-1} \to x_i$ for $i = n\ell+1, \ldots, n(\ell+1)$. By observing vertices x_0 and x_{n^2} we find that for even n we have

$$\mathrm{diam}_{\min}(G_n) \ge \frac{n}{2} + \frac{n}{2}\cdot n = \frac{1}{2}(n^2+n)$$

where the optimal orientation that minimizes the diameter (more precisely, minimizes the distances between x_0 and x_{n^2}) is the orientation where one half of edges $x_{n\ell}x_{n(\ell+1)}$ is directed $x_{n\ell} \to x_{n(\ell+1)}$ and the other half is directed $x_{n\ell} \leftarrow x_{n(\ell+1)}$. The optimal orientation that minimizes the weak diameter is the orientation where for every ℓ, $x_{n\ell} \to x_{n(\ell+1)}$. Therefore

$$\mathrm{wdiam}_{\min}(G_n) \le 2(n-1) + n = 3n-2$$

which proves the claim.

The parameter $\mathrm{diam}_{\min}(G)$ was studied from theoretical and practical points of view. It is important in traffic control problems (see [13]), when two-way streets are turned into one-way streets to achieve a better traffic flow. The objective is to minimize the longest distance when doing this. A general bound for $\mathrm{diam}_{\min}(G)$ was obtained in [3] (see also [1]) where the following result was proved.

Theorem 1.1 ([3]). *For every bridgeless connected graph G of radius r,* $\mathrm{diam}_{\min}(G) \le 2r^2 + 2r$.

In this article we consider the diameter of orientations of Cartesian products of graphs. The *Cartesian product* of graphs G and H is the graph, denoted as $G\Box H$, with vertex set $V(G\Box H) = V(G) \times V(H)$, where vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G\Box H$ if and only if $x_1x_2 \in E(G)$ and $y_1 = y_2$, or $x_1 = x_2$ and $y_1y_2 \in E(H)$. Several results on distances in Cartesian products of graphs are given in [4]. For $y \in V(H)$ the *G-layer* G_y is the set $G_y = \{(x, y) \mid x \in V(G)\}$. Analogously we define H-layers. The diameter of orientations of Cartesian products (when one of the factors is bipartite) was addressed by Koh and Tay in [9]. The same authors proved in [11] that Cartesian products of trees admit orientations such that the diameter of the orientation is equal to the diameter of the underlying undirected graph.

Theorem 1.2 ([11]). *If T_1 and T_2 are trees with diameters at least 4, then*

$$\mathrm{diam}_{\min}(T_1\Box T_2) = \mathrm{diam}(T_1\Box T_2) .$$

They also considered orientations of $K_m\Box K_n$, $K_m\Box P_n$, $P_m\Box C_n$ and $K_m\Box C_n$ (see [6–8,10]). In [12] it was proved that $\mathrm{diam}_{\min}(C_m\Box C_n) = \mathrm{diam}(C_m\Box C_n)$ for $m, n \ge 6$.

Some related problems, like strong diameter and strong radius of Cartesian products are studied in [2] and [5], where the strong radius of Cartesian products is exactly determined, and an upper bound for the strong diameter is given.

For any connected bridgeless graphs G and H we have

$$\mathrm{diam}_{\min}(G\Box H) \le \mathrm{diam}_{\min}(G) + \mathrm{diam}_{\min}(H) .$$

To see this let D_G and D_H be orientations of G and H, respectively, such that $\mathrm{diam}(D_G) = \mathrm{diam}_{\min}(G)$ and $\mathrm{diam}(D_H) = \mathrm{diam}_{\min}(H)$. Since G-layers are isomorphic to G we can give the edges in G-layers the orientation D_G, and similarly we give the edges in H-layers the orientation D_H. The diameter of the obtained orientation is at most $\mathrm{diam}(D_G) + \mathrm{diam}(D_H)$. The objective of this paper is to improve the bound above, and to give a bound for $\mathrm{diam}_{\min}(G\Box H)$ in terms of $\mathrm{wdiam}_{\min}(G)$ and $\mathrm{wdiam}_{\min}(H)$.