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Note

Strict chordal digraphs viewed as graphs with distinguished edges

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ABSTRACT

P. Hell and C. Hernández-Cruz recently defined new directed graph analogs of the traditional concepts of chordal and split graphs. This paper will provide additional evidence of the naturalness of their digraph analogs using characterizations in terms of undirected graphs with distinguished edges, most notably using standard clique tree representations from chordal graph theory.

1. Introduction and $\mathcal{D}$-chordal graphs

Let $\vec{G}$ denote a digraph that is an orientation of a graph G in which each edge xy of G becomes a uni-directed arc ($x \to y$ or $x \leftarrow y$) or a bi-directed arc ($x \leftrightarrow y$) of $\vec{G}$. Suppose $V(\vec{G})$ is ordered as $\langle v_1, \dots, v_n \rangle$ and, when $1 \le i \le n$, let $G_i = G[\{v_i, \dots, v_n\}]$, where $G[S]$ denotes the subgraph induced by $S \subseteq V(G)$. After discussing other ways that graph theoretic concepts have been expanded to digraphs, P. Hell and C. Hernández-Cruz [2] define $\vec{G}$ to be a *strict chordal digraph* if, for all i, every two vertices in the neighborhood $N_i(v_i)$ of v_i in G_i are joined by a bi-directed arc of $\vec{G}$ (although other arcs of $\vec{G}$ can also be bi-directed). In particular, [2] presents a minimal set of digraphs that cannot be induced in strict chordal digraphs and considers the algorithmic consequences of this set of obstructions. (The discussion below, however, will be essentially independent of the results in [2].)

Somewhat unexpectedly, this definition of strict chordal digraphs does not involve the directions of the uni-directed arcs of $\vec{G}$; all that matters is that the arcs between vertices in each $N_i(v_i)$ are bi-directed. Motivated by this, the present paper will consider how strict chordal digraphs translate into undirected graphs with sets of distinguished edges and will characterize the resulting graphs using traditional concepts of chordal graph theory. This may provide additional perspective on the concepts introduced in [2].

A graph G is a *chordal graph* if $V(G)$ can be ordered as $\langle v_1, \dots, v_n \rangle$ with the neighborhood $N_i(v_i)$ of each v_i in the subgraph $G_i = G[\{v_i, \dots, v_n\}]$ always inducing a complete subgraph of G (possibly a subgraph with empty vertex set). Such a sequence $\langle v_1, \dots, v_n \rangle$ is called a *perfect elimination ordering* of G. Every induced subgraph G' of a chordal graph G is also chordal, since the subsequence of the perfect elimination ordering of G that consists of vertices of G' will be a perfect elimination ordering of G'. See [1,3,7] for detailed discussions of chordal graphs, including the classic result that a graph is chordal if and only if every cycle of length 4 or more has a chord—in other words, if every cycle long enough to have a chord does have a chord.

A vertex v is a *simplicial vertex* of G if $N(v)$ induces a complete subgraph. Thus, $\langle v_1, \dots, v_n \rangle$ is a perfect elimination ordering of G if and only if each v_i is a simplicial vertex of G_i, and a graph is chordal if and only if every induced subgraph has a

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Fig. 1. A $\mathcal{D}$-chordal graph (on the left) and a non-$\mathcal{D}$-chordal graph.

simplicial vertex. A *maxclique* Q of G is an inclusion-maximal set of pairwise-adjacent vertices; equivalently, $Q \subseteq V(G)$ induces a maximal complete subgraph. Thus, v is a simplicial vertex of G if and only if $N(v) \cup \{v\}$ is the unique maxclique of G that contains v.

Define a graph G with a set $\mathcal{D} \subseteq E(G)$ of distinguished edges to be a *$\mathcal{D}$-chordal graph* if G is chordal with a perfect elimination ordering $\langle v_1, \ldots, v_n\rangle$ such that each $N_i(v_i)$ induces a complete subgraph whose edges are all in $\mathcal{D}$; call such a sequence $\langle v_1, \ldots, v_n\rangle$ a *$\mathcal{D}$-elimination ordering* of G. Define v to be a *$\mathcal{D}$-simplicial vertex* of G if $N(v)$ induces a complete subgraph whose edges are all in $\mathcal{D}$. Thus, $\langle v_1, \ldots, v_n\rangle$ is a $\mathcal{D}$-elimination ordering of G if and only if each v_i is a $\mathcal{D}$-simplicial vertex of G_i. Furthermore, every induced subgraph of a $\mathcal{D}$-chordal graph is $\mathcal{D}$-chordal, and a graph is $\mathcal{D}$-chordal if and only if every induced subgraph has a $\mathcal{D}$-simplicial vertex.

Notice that if $\mathcal{D} = E(G)$, then G being $\mathcal{D}$-chordal is equivalent to G being chordal. But not all of the edges of G have to be in $\mathcal{D}$ in order for a chordal graph G to be $\mathcal{D}$-chordal; for instance, an edge that has a simplicial endpoint is never an edge of a $G[N_i(v_i)]$ and so is not required to be in $\mathcal{D}$. Corollary 4 will characterize the edges that are required to belong to $\mathcal{D}$ in this sense. At the other extreme, if $\mathcal{D} = \emptyset$, then G being $\mathcal{D}$-chordal is equivalent to G being a forest (and so to not having any $|N_i(v_i)| > 1$).

The graph on the left in Fig. 1 is $\mathcal{D}$-chordal with a $\mathcal{D}$-elimination ordering $\langle d, e, c, a, b, f\rangle$ where—as in all the figures in this paper—edges in $\mathcal{D}$ are shown as solid edges, with edges in $E(G) - \mathcal{D}$ as dotted edges. The graph on the right is not $\mathcal{D}$-chordal (in fact, none of its vertices are $\mathcal{D}$-simplicial).

The definition of strict chordal digraphs at the beginning of this section, from [2], and the new definition of $\mathcal{D}$-chordal graphs correspond as follows:

> A graph G with a set $\mathcal{D}$ of distinguished edges is a $\mathcal{D}$-chordal graph if and only if every orientation $\vec{G}$ in which the edges of $\mathcal{D}$ become bi-directed arcs is a strict chordal digraph.

Lemma 1. *Every cycle of length* 4 *or more in a* $\mathcal{D}$*-chordal graph has a chord in* $\mathcal{D}$*.*

Proof. Suppose G is $\mathcal{D}$-chordal with a cycle C of length 4 or more. Thus the $\mathcal{D}$-chordal subgraph $G[V(C)]$ has a $\mathcal{D}$-simplicial vertex v and distinct edges uv and vw of C. This makes uw a chord of C that is in $\mathcal{D}$. □

Lemma 2. *If* $Q \subseteq V(G)$ *induces a complete subgraph of a* $\mathcal{D}$*-chordal graph* G*, then* Q *contains* $|Q| - 1$ *vertices that pairwise induce edges in* $\mathcal{D}$*.*

Proof. If G is $\mathcal{D}$-chordal and $Q \subseteq V(G)$ has $G[Q]$ complete and $|Q| = q$, then $G[Q]$ is $\mathcal{D}$-chordal with a $\mathcal{D}$-elimination ordering $\langle v_1, \ldots, v_q\rangle$, and so $S = N(v_1) \cap Q$ has $|S| = q - 1$ where every two vertices of S are adjacent in $\mathcal{D}$. □

In the proof of Lemma 2, it is easy to see that every such Q contains either exactly one such S (namely, $N(v) \cap Q$ with v the unique $\mathcal{D}$-simplicial vertex of $G[Q]$) *or* exactly two such S (namely, $N(v) \cap Q$ and $N(v') \cap Q$ with vv' the unique edge of $G[Q]$ that is not in $\mathcal{D}$) *or* exactly $|Q|$ such S (namely, every $N(v) \cap Q$ with $v \in Q$ and $E(G[Q]) \subseteq \mathcal{D}$).

The non-$\mathcal{D}$-chordal graph G on the right in Fig. 1 shows that the converses to Lemmas 1 and 2 both fail.

2. Characterizing $\mathcal{D}$-chordal graphs

As an alternative to the vertex elimination ordering approach used in [2], clique tree representations offer a different, yet closely related, standard approach to chordal graphs; see [1,3,7]. A *clique tree* T for a graph G is a tree T whose vertices are precisely the maxcliques of G such that, for each $v \in V(G)$, the vertices of T that contain v induce a subtree of T. A graph is chordal if and only if it has a clique tree and, using this point of view, chordal graphs can be characterized as the intersection graphs of subtrees of trees. See [5] for linear-time algorithms for constructing clique trees.

From here on we will call the vertices of a clique tree T the *nodes* of T to lessen confusion between vertices of T and of G. Each leaf node of a clique tree T for G contains at least one simplicial vertex of G (although non-leaf nodes can also contain simplicial vertices).

Fig. 2 shows a $\mathcal{D}$-chordal graph G, one $\mathcal{D}$-elimination ordering for which is $\langle a, h, b, g, i, j, c, d, e, f\rangle$, along with a clique tree T of G that happens to be the unique clique tree for this particular G. There are seven simplicial vertices of G (a, b, d, e, g, h, and i, each in a unique node of T), three non-simplicial vertices (c, f, and j, each in three nodes that induce a subtree of T), and only two $\mathcal{D}$-simplicial vertices (a and h). This example will be used to illustrate the proof of Theorem 3, which is the $\mathcal{D}$-chordal version of the clique tree characterization of chordal graphs mentioned above.