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Critical and maximum independent sets of a graph

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ABSTRACT

Let G be a simple graph with vertex set $V(G)$. A set $A \subseteq V(G)$ is *independent* if no two vertices from A are adjacent. If $\alpha(G) + \mu(G) = |V(G)|$, then G is called a König–Egerváry graph (Deming, 1979; Sterboul, 1979), where $\alpha(G)$ is the size of a maximum independent set and $\mu(G)$ stands for the cardinality of a largest matching in G.

The number $d(X) = |X| - |N(X)|$ is the *difference* of $X \subseteq V(G)$, and a set $A \subseteq V(G)$ is *critical* if $d(A) = \max\{d(X) : X \subseteq V(G)\}$ (Zhang, 1990).

In this paper, we present various connections between unions and intersections of maximum and/or critical independent sets of a graph, which lead to new characterizations of König–Egerváry graphs.

1. Introduction

Throughout this paper G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X. By $G - W$ we mean either the subgraph $G[V(G) - W]$, if $W \subseteq V(G)$, or the subgraph obtained by deleting the edge set W, for $W \subseteq E(G)$. The *neighborhood* $N(v)$ of $v \in V(G)$ is the set $\{w : w \in V(G)$ *and* $vw \in E(G)\}$. The *neighborhood* $N(A)$ of $A \subseteq V(G)$ is $\{v \in V(G) : N(v) \cap A \neq \emptyset\}$. For a family of sets Λ we will write $\bigcup \Lambda$ instead of $\bigcup_{S\in\Lambda} S$, and $\bigcap \Lambda$ instead of $\bigcap_{S\in\Lambda} S$.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of G. An independent set of maximum size is a *maximum independent set* of G, and $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$.

A *matching* is a set M of pairwise non-incident edges of G. If every vertex of a set A is an endpoint of an edge $e \in M$, while the other endpoint of e belongs to some set B, disjoint from A, we say that M is a matching from A into B, or A is matched into B by M. In other words, M may be interpreted as an injection from the set A into the set B. If $2|M| = V(G)$, then M is a *perfect matching*. By $\mu(G)$ is denoted the size of a largest matching.

For $X \subseteq V(G)$, the number $|X| - |N(X)|$ is the *difference* of X, denoted $d(X)$. The *critical difference* $d(G)$ of the graph G is $\max\{d(X) : X \subseteq V(G)\}$. The number $\max\{d(I) : I \in \mathrm{Ind}(G)\}$ is the *critical independence difference* of G, denoted $id(G)$. Clearly, $d(G) \geq id(G)$. It was shown in [43] that $d(G) = id(G)$ holds for every graph G. If $A \subseteq V(G)$ and $d(A) = d(G)$, then A is a *critical set* [43].

Theorem 1.1. *For every critical independent set A, the following hold:*
(i) [7] A is included in some maximum independent set;
(ii) [20] there is a matching from $N(A)$ into A.

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Critical independent sets define an important area of research due to their close connections to the well-known **NP**-hard problem of finding a maximum independent set [2,11,15,22,31,35] . Actually, every critical independent set is contained in a maximum critical independent set, and a maximum critical independent set can be found in polynomial time [20]. This leads to a practically efficient way of approximating the size of the independence number [42], since every critical independent set is a subset of a maximum independent set [7]. This technique of finding the independence number based on maximum critical independent sets was shown to be highly effective in practice for medium-sized networks [41]. Moreover, there exists a non-trivial partition of G into two subgraphs with a polynomial algorithm returning a maximum independent set of at least one of these subgraphs [21].

It is known that $\alpha(G) + \mu(G) \leq |V(G)|$ holds for each graph. If $\alpha(G) + \mu(G) = |V(G)|$, then G is a *König–Egerváry graph* [12,40]. A vertex cover of G is a set A of vertices such that A contains at least one endpoint of every edge of G. The vertex cover number $\tau(G)$ is the minimum size of a vertex cover. It is known that $\alpha(G) + \tau(G) = |V(G)|$. Hence, G is a König–Egerváry graph if and only if $V(G) - S$ is a minimum vertex cover for every maximum independent set S, i.e., if and only if $\mu(G) = \tau(G)$.

For example, every bipartite graph is a König–Egerváry graph as well [14,18]. These graphs are well-studied in the literature [4,10,17,34,37–39]. In [36] König–Egerváry graphs with perfect matchings were characterized by a family of forbidden subgraphs. Further, it was extended to general König–Egerváry graphs in [5,19]. A characterization of König–Egerváry graphs in terms of spanning trees was obtained in [8]. Different properties of matchings make possible to outline König–Egerváry graphs as well [17,30].

Critical independent sets have a particularly nice structure in König–Egerváry graphs. For instance, we make use of the following.

Theorem 1.2 (*[21,26]*). *For a graph G, the following assertions are equivalent:*

(i) G is a König–Egerváry graph;

(ii) there exists some maximum independent set which is critical;

(iii) each of its maximum independent sets is critical.

König–Egerváry graphs may also be described as graphs admitting a decomposition into a maximum independent set S and a subgraph H such that $V(G) = S \cup V(H)$, and there is a matching from $V(H)$ into S [25].

If A is an independent set of G such that there exists a matching from $N(A)$ into A, then $\{A, N(A)\}$ is called a crown structure of G [1]. Crown structures are known to be important tools for fixed parameter tractable problems [9].

In our language, a crown structure may be interpreted as a König–Egerváry subgraph, which is, in turn, closely related to the notion of a critical independent set. For instance, it is known that the vertex cover problem is **NP**-complete, while in the context of parameterized complexity this problem is fixed parameter tractable [13]: the size of the vertex cover problem can be substantially reduced by deleting the vertices of $A \cup N(A)$ and their adjacent edges, where $\{A, N(A)\}$ is a crown structure.

The paper is organized as follows. Section 2 contains a number of results we need later. In Section 3 we present several properties of unions and intersections of maximum and/or critical independent sets leading to new characterizations of König–Egerváry graphs, which we describe in Section 4.

2. Preliminaries

In the early 1980s, Berge found the following characterization of maximum independent sets, which will be further required in the paper.

Theorem 2.1 (*[3]*). *An independent set S is maximum if and only if for every independent set A disjoint from S there exists a matching from A into S.*

Theorem 2.1 was extended to interconnections between a family of maximum independent sets and an independent set as follows.

Lemma 2.2 (*Matching Lemma*). *[33] If* $A \in \mathrm{Ind}(G)$, $\Lambda \subseteq \Omega(G)$, *and* $|\Lambda| \geq 1$, *then there exists a matching from* $A - \bigcap \Lambda$ *into* $\bigcup \Lambda - A$.

Recall that $\mathrm{core}(G) = \bigcap\{S : S \in \Omega(G)\}$ [23], $\mathrm{corona}(G) = \bigcup\{S : S \in \Omega(G)\}$ [6], and $\ker(G) = \bigcap\{A : A$ is critical and $A \in \mathrm{Ind}(G)\}$ [28]. A number of properties of $\ker(G)$ and $\mathrm{core}(G)$ can be found in [27,31,32].

Theorem 2.3. *For a graph G, the following assertions are true:*

(i) [28] $\ker(G) \subseteq \mathrm{core}(G)$, *and* $\ker(G)$ *is the unique minimal independent critical set;*

(ii) [28] if A and B are critical in G, then $A \cup B$ *and* $A \cap B$ *are critical as well;*

(iii) [16] if $\Lambda \subseteq \Omega(G)$, *and* $|\Lambda| \geq 1$, *then* $2\alpha(G) \leq \left|\bigcap \Lambda\right| + \left|\bigcup \Lambda\right| \leq 2(|V(G)| - \mu(G))$;

(iv) [16] if $\Lambda_1 \subseteq \Lambda_2 \subseteq \Omega(G)$, *then* $\left|\bigcup \Lambda_1\right| + \left|\bigcap \Lambda_1\right| \leq \left|\bigcup \Lambda_2\right| + \left|\bigcap \Lambda_2\right|$.

Let $\Omega(G)$ denote the family of all maximum independent sets of G. If $\Lambda \subseteq \Omega(G)$, $|\Lambda| \geq 1$, and $\left|\bigcap \Lambda\right| + \left|\bigcup \Lambda\right| = 2\alpha(G)$, then Λ is a *König–Egerváry collection* in G [16]. Clearly, each family $\Lambda \subseteq \Omega(G)$ satisfying $|\Lambda| \in \{1, 2\}$ is a König–Egerváry collection.