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Connectivity, diameter, minimal degree, independence number and the eccentric distance sum of graphs[☆]

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ABSTRACT

The eccentric distance sum (EDS) of a connected graph G is defined as $\xi^d(G) = \sum_{\{u,v\}\subseteq V_G} (\varepsilon_G(u) + \varepsilon_G(v))d_G(u, v)$, where $\varepsilon_G(\cdot)$ is the eccentricity of the corresponding vertex and $d_G(u, v)$ is the distance between u and v in G. In this paper, some extremal problems on the EDS of an n-vertex graph with respect to other two graph parameters are studied. Firstly, k-connected (bipartite) graphs with given diameter having the minimum EDS are characterized. Secondly, sharp lower bound on the EDS of connected graphs with given connectivity and minimum degree is determined. Lastly sharp lower bound on the EDS of connected graphs with given connectivity and independence number is determined as well.

1. Introduction

Throughout this paper, we only consider simple and connected graphs. Let $G = (V_G, E_G)$, where V_G is the vertex set and E_G is the edge set. We call $|V_G|$ the *order* and $|E_G|$ the *size* of G. The *distance* $d_G(u, v)$ between two vertices u, v of G is the length of a shortest u–v path in G. The *eccentricity* of a vertex v, written $\varepsilon_G(v)$ (or $\varepsilon(v)$ for short), is $\max_{u\in V_G} d(u, v)$. The *diameter* of G, written by diam$G$, is $\max_{u\in V_G} \varepsilon(u)$. Unless otherwise stated, we follow the traditional notations and terminologies (see, for instance, [1,14]).

Let G be a graph with $u, v \in V_G$. Then $G - v$ (resp. $G - uv$) denote the graph obtained from G by deleting vertex $v \in V_G$ (resp. edge $uv \in E_G$). We may extend naturally this notation if more than one vertex or edge is deleted. Similarly, $G + xy$ is obtained from G by adding an edge $xy \notin E_G$.

For a vertex subset S of V_G, denote by $G[S]$ the subgraph induced by S. Denote by P_n, K_n and $K_{r,n-r}$ the path, complete graph and complete bipartite graph on n vertices, respectively.

Let $N_G(v)$ denote the set of all vertices being adjacent to v. The degree $d(v)$ of a vertex v is equal to $|N_G(v)|$. The number $\Delta(G) := \max\{d(v) : v \in V_G\}$ is the maximum degree of G, whereas $\delta(G) := \min\{d(v) : v \in V_G\}$ is the minimum degree of G. For a real number x, denote by $\lfloor x\rfloor$ the greatest integer $\leqslant x$, and by $\lceil x\rceil$ the least integer $\geqslant x$.

A *vertex cut* of a connected graph G is a vertex subset S of V_G such that $G - S$ has more than one components. The *connectivity* of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex.

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A graph is *k-connected* if its connectivity is at least k. An *independent set* in a graph is a set of pairwise nonadjacent vertices. The *independence number* of G, written $\alpha(G)$, is the maximum size of an independent set of vertices.

The *join* $H_1 \oplus H_2$ of two vertex disjoint graphs H_1 and H_2 is the graph consisting of the union $H_1 \cup H_2$, together with all edges of the type st, where $s \in V_{H_1}$ and $t \in V_{H_2}$. For $m \geqslant 3$ vertex disjoint graphs $H_1, H_2, \ldots, H_m$, the *sequential join* $H_1 \oplus H_2 \oplus \cdots \oplus H_m$ is the graph $(H_1 \oplus H_2) \cup (H_2 \oplus H_3) \cup \cdots \cup (H_{m-1} \oplus H_m)$. The sequential join of m disjoint copies of a graph H will be denoted by $[m]H$, the union of r disjoint copies of H will be denoted by rH, while $[s]H_1 \oplus H_2 \oplus [t]H_3$ will denote the sequential join $\overbrace{H_1 \oplus H_1 \oplus \cdots \oplus H_1}^{s} \oplus H_2 \oplus \overbrace{H_3 \oplus H_3 \oplus \cdots \oplus H_3}^{t}$.

In 2002, Gupta, Singh and Madan [3] introduced the *eccentric distance sum* $\xi^d(G)$ of a graph G as

$$\xi^d(G) = \sum_{\{u,v\} \subseteq V_G} (\varepsilon_G(u) + \varepsilon_G(v)) d(u, v),$$

which is equivalent to

$$\xi^d(G) = \sum_{v \in V_G} \varepsilon_G(v) D_G(v),$$

where $D_G(v) = \sum_{u \in V_G} d(u, v)$. This graph invariant not only presents a vast potential for structure activity and property relationships, but also offers precious leads for the advancement of safe and potent curative agents of multiple nature as well. Comparatively, the eccentric distance sum offers excellent prediction than the Wiener's index due to the contribution of eccentricity. One may be referred to [3] for more details.

It is interesting to determine the extremal eccentric distance sum of graphs with given structure properties, which attracts much attention in the literature. Mukungunugwa and Mukwembi [13] determined the asymptotic upper bounds for the EDS of graphs according to its order and minimal degree. Yu, Feng and Ilić [15] determined the n-vertex tree of given diameter having the minimal eccentric distance sum. Ilić et al., [6] established various lower and upper bounds for the EDS in terms of some other graph invariants. Hua et al., [4] established some further bounds for the EDS with respect to some graphs parameters. Li et al., [10] determined all trees with given matching number having the minimal and superminimal eccentric distance sums. Geng, Li and Zhang [2] characterized the n-vertex trees with domination number γ having the minimal EDS and identified n-vertex trees with domination number γ satisfying $n = k\gamma$ having the maximal EDS with $k = 2, n/3, n/2$. Miao et al., [11] characterized the graphs with maximal EDS among n-vertex trees with given parameters including the maximum degree, domination number 3, independence number and matching number. Li, Wu and Sun [9] determined the graphs with the minimum EDS among n-vertex connected bipartite graphs with some given parameters such as matching number, diameter, vertex connectivity.

Very recently, Miao et al., [12] identified the extremal tree among n-vertex trees with domination number γ with $4 \leqslant \gamma \leqslant \lceil n/3 \rceil$ having the maximal EDS. Li and Wu [8] studied the relationships between the EDS and graph parameters such as the order, the size, the diameter and the connectivity is discussed: Sharp lower bounds on the EDS of n-vertex connected triangle free graphs and n-vertex connected graphs of size m were given. A sharp lower bound on the EDS of an n-vertex connected graph with given diameter was determined. A sharp upper bound on the EDS of an n-vertex graph with even connectivity was determined. For more results and progress on the eccentric distance sum of graphs, one may be referred to the survey [7] and the references therein.

Our paper is motivated directly from [5,16]. In the study of the eccentric distance sum, one generally considers the n-vertex graph with only one other graph parameter. It is natural and interesting for us to study the extremal property on the EDS by focusing on the n-vertex graph with some other two parameters. In mathematical literature, more and more researchers study the relationship between one graph parameter and other two invariants; see [5,16] for example.

This paper is organized as follows. In Section 2, we determine the graph which attains the minimum EDS among all k-connected graphs of given diameter; In Section 3, we identify k-connected bipartite graph with given diameter having the minimum EDS; In Section 4, we establish the sharp lower bound on the EDS of n-vertex graphs with given connectivity and minimum degree. In the last section, we identify all the extremal graphs with the minimum EDS in the class of all n-vertex graphs with given connectivity and independence number.

Further on, we need the following lemmas.

Lemma 1.1 ([15]). *Let G be a connected graph of order n and $G \ncong K_n$. Then $\xi^d(G) > \xi^d(G + e)$ for each edge $e \notin E_G$.*

Lemma 1.2 ([13]). *Let G be an n-vertex connected graph with the minimum degree $\delta \geqslant 2$. Then*

$$\xi^d(G) \leqslant \frac{3 \cdot 5^2}{2^5 \cdot (\delta + 1)^2} n^4 + O(n^3).$$

Moreover, for a fixed δ, this bound is asymptotically sharp.

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