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Wiener index and Harary index on Hamilton-connected graphs with large minimum degree☆

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ABSTRACT

In this paper, we give some sufficient conditions for connected graphs with given minimum degree to be Hamilton-connected and traceable from every vertex in terms of Wiener index and Harary index. We also prove sufficient conditions on Harary index of $\overline{G}$ for G to be Hamilton-connected and traceable from every vertex with given minimum degree.

1. Introduction

Let $G = (V(G), E(G))$ be a graph with $|V(G)| = n$ and $|E(G)| = e(G)$. We use $N_G(v)$ to denote the set of vertices adjacent to v in G. The degree of v is denoted by $d_G(v) = |N_G(v)|$. Denote by $\delta(G)$ the minimum degree of G. The distance between two vertices u and v in G, i.e., the length of a shortest path connecting u and v in G, is denoted by $d_G(u, v)$. Denote by $diam(G)$ the diameter of G. If the graph G is clear under the context, we will drop the subscript of G. The complement $\overline{G}$ of G is the graph on $V(G)$ with edge set $[V(G)]^2 \setminus E(G)$. For two graphs G and H, we denote the union of G and H by $G+H$, and the join of G and H by $G \vee H$. The union of k disjoint copies of the same graph G is denoted by kG. For terminology and notation not defined but used, we refer the reader to [1].

A cycle (path) is called Hamilton cycle (path) if it contains every vertex of G. A graph is called Hamiltonian (traceable) if it contains a Hamilton cycle (path). A graph is called Hamilton-connected if every two vertices are connected by a Hamilton path. A graph is called traceable from a vertex x if it has a Hamilton x-path.

The Wiener index, denoted by $W(G)$, was introduced in 1947 by Wiener [14]. It is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v).$$

The Harary index, denoted by $H(G)$, has been introduced by Ivanciuc et al. [8] and Plavšić et al. [13]. It is defined as

$$H(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{d_G(u, v)}.$$

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Note that, in any disconnected graph G, the distance is infinite of any two vertices from two distinct components. Therefore its reciprocal can be viewed as 0. Thus, we can define validly the Harary index of disconnected graph G as follows:

$$H(G)=\sum_{i=1}^{k} H(G_i),$$

where $G_1, G_2, \ldots, G_k$ are the components of G. The readers can refer to [2–4,9,12,15,16,18] to know more about these two indices.

The problem of determining the Hamiltonicity of graphs is NP-hard. It is meaningful to find sufficient conditions for a graph to be Hamiltonian, traceable or Hamilton-connected. Hua et al. [7] gave a sufficient condition for a graph to be traceable in terms of Harary index. Yang [17] gave a sufficient condition on Wiener index for a graph to be traceable. There are some errors in proofs of their papers. In [10,11], Liu et al. corrected their results respectively and also proposed some sufficient conditions for Hamiltonian graphs on Wiener index and Harary index. By imposing minimum degree, Hua et al. [6] obtained sufficient conditions for Hamiltonicity and traceability of connected graphs and of connected balanced bipartite graphs in terms of Wiener index and Harary index. The other related results on this aspect can be found in [5,20].

The rest paper is organized as follows. In Section 2, we will present two structural results which are crucial for the proofs of main results. In Section 3, we will give two sufficient conditions for a connected graph with given minimum degree to be Hamilton-connected or traceable from every vertex in terms of Wiener index and Harary index. These two results determine the value of $\min W(G)$ and $\max H(G)$ among all graphs G that are not Hamilton-connected (or not traceable from every vertex) of order n with $\delta(G)\geq k$, respectively. In Section 4, we obtain some sufficient conditions for Hamilton-connectivity of a graph G on Harary index of $\overline{G}$ with given minimum degree.

2. Preliminaries

In this section, we mainly give two structural results and one useful lemma.

Firstly, for $n\geq 5$ and $1\leq k\leq n/2$, we define:

$$S_n^k=K_k\vee(K_{n-(2k-1)}+(k-1)K_1) \text{ and } T_n^k=K_2\vee(K_{n-(k+1)}+K_{k-1}).$$

The graph S_n^k is obtained from K_{n-k+1} and $(k-1)K_1$ by connecting all vertices of $(k-1)K_1$ to all vertices of a k-subset of K_{n-k+1}. The graph T_n^k is obtained from K_{n-k+1} and K_{k+1} by identifying two vertices. Note that $S_n^2=T_n^2$. Denote by $\underline{S}_n^k$ and $\underline{T}_n^k$ the graphs obtained from S_{n+1}^{k+1} and T_{n+1}^{k+1}, respectively, by deleting one vertex of degree n, i.e., for $0\leq k\leq (n+1)/2-1$,

$$\underline{S}_n^k=K_k\vee(K_{n-2k}+kK_1) \text{ and } \underline{T}_n^k=K_1\vee(K_{n-k-1}+K_k).$$

Note that $\underline{S}_n^1=\underline{T}_n^1$.

Lemma 2.1 (*Exercise 18.1.6 on Page 474 of [1]*). *Let G be a graph. Then G is traceable from every vertex if and only if $G\vee K_1$ is Hamilton-connected.*

Our proofs of main results depend on the following two theorems. The first one is due to [19].

Theorem 2.2 (*[19]*). *Let G be a graph of order $n\geq 11k$ with minimum degree $\delta(G)\geq k$, where $k\geq 2$. If*

$$e(G)>\binom{n-k}{2}+k(k+1),$$

then G is Hamilton-connected unless $G\subseteq S_n^k$ or T_n^k.

Theorem 2.3. *Let G be a graph of order $n\geq 11k+10$ with minimum degree $\delta(G)\geq k$, where $k\geq 1$. If*

$$e(G)>\binom{n-k-1}{2}+(k+1)^2,$$

then G is traceable from every vertex unless $G\subseteq \underline{S}_n^k$ or $\underline{T}_n^k$.

Proof. Let $G'=G\vee K_1$. Then we have $|V(G')|=n+1\geq 11(k+1)$, $\delta(G')\geq k+1$ and

$$e(G')=e(G)+n>\binom{n-k-1}{2}+(k+1)^2+n=\binom{n-k}{2}+(k+1)(k+2).$$

By Theorem 2.2, G' is Hamilton-connected unless $G'\subseteq S_{n+1}^{k+1}$ or T_{n+1}^{k+1}. From Lemma 2.1, we know that G is traceable from every vertex if and only if G' is Hamilton-connected. Hence, G is traceable from every vertex unless $G\subseteq \underline{S}_n^k$ or $\underline{T}_n^k$.

The proof is complete. ■