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Matchings in graphs and groups

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ABSTRACT

Berge's Lemma says that for each independent set S and maximum independent set X, there is a matching from $S - X$ into $X - S$, namely, a function of $S - X$ into $X - S$ such that $(s, f(s))$ is an edge for each $s \in S - X$. Levit and Mandrescu prove A Set and Collection Lemma. It is a strengthening of Berge's Lemma, by which one can obtain a matching $M : S - \bigcap \Gamma \to \bigcup \Gamma - S$, where S is an independent set and Γ is a collection of maximum independent sets. Jarden, Levit and Mandrescu invoke new inequalities from A Set and Collection Lemma and yield a new characterization of König–Egerváry graphs.

In the current paper, we study Berge systems (collections of vertex sets, where the conclusion of Berge's Lemma hold). The work is divided into two parts: the general theory of Berge systems and Berge systems associated with groups (or more precisely, in non-commuting graphs).

We first show that there exists many Berge systems. The notion of an ideal is central in several branches of mathematics. We define 'a graph ideal', which is a natural generalization of 'an ideal' in the context of graph theory. We show that every graph ideal carries a Berge system.

We prove a sufficient condition for a Berge system to be a multi-matching system (collections where the conclusion of a Set and Collection Lemma holds). We get new inequalities in multi-matching systems.

Here, we do a turn about from the general theory of Berge systems into the study of Berge systems in non-commuting graphs. We prove that for every group G, if A and B are maximal abelian subsets of $\langle A, B\rangle_G$ and $|A| \le |B|$, then there is a non-commutative matching of $A - B$ into $B - A$. It means that every non-commuting graph carries a Berge system. Moreover, we apply the sufficient condition to get a multi-matching system in the context of non-commuting graphs and invoke new inequalities in group theory.

Surprisingly, the new results concerning groups, do not hold for rings.

1. Introduction

Throughout this paper $\mathbb{G}$ is a finite simple graph with vertex set $V(\mathbb{G})$ and edge set $E(\mathbb{G})$. If $X \subseteq V(\mathbb{G})$, then $\mathbb{G}[X]$ is the subgraph of $\mathbb{G}$ induced by X. By $\mathbb{G} - W$ we mean either the subgraph $\mathbb{G}[V(\mathbb{G}) - W]$, if $W \subseteq V(\mathbb{G})$, or the subgraph obtained by deleting the edge set W, for $W \subseteq E(\mathbb{G})$. In either case, we use $\mathbb{G} - w$, whenever $W = \{w\}$. If $A, B \subseteq V(\mathbb{G})$, then (A, B) stands for the set $\{ab : a \in A, b \in B, ab \in E(\mathbb{G})\}$.

The *neighborhood* $N(v)$ of a vertex $v \in V(\mathbb{G})$ is the set $\{w : w \in V(\mathbb{G}) \text{ and } vw \in E(\mathbb{G})\}$; in order to avoid ambiguity, we use also $N_{\mathbb{G}}(v)$ instead of $N(v)$. The *neighborhood* $N(A)$ of $A \subseteq V(\mathbb{G})$ is $\{v \in V(\mathbb{G}) : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$.

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A set $S \subseteq V(\mathbb{G})$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(\mathbb{G})$ we mean the family of all independent sets of $\mathbb{G}$. An independent set of maximum size is a *maximum independent set* of $\mathbb{G}$, and $\alpha(\mathbb{G}) = \max\{|S| : S \in \mathrm{Ind}(\mathbb{G})\}$.

A *matching* is a set M of pairwise non-incident edges of $\mathbb{G}$. Abusing notation, we say that $M : A \to B$ is a matching, when $M : A \to B$ is a function and each pair $(a, M(a))$ is an edge. If $A \subseteq V(\mathbb{G})$, then $M(A)$ is the set of all vertices matched by M with vertices belonging to A. A matching of maximum cardinality, denoted $\mu(\mathbb{G})$, is a *maximum matching*. For every matching M, we denote the set of all vertices that M saturates by $V(M)$, and by $M(x)$ we denote the vertex y satisfying $xy \in M$.

Lemma 1.1 ([3,4]). *Let S be an independent set in $\mathbb{G}$ and let X be an independent set in $\mathbb{G}$ of maximum cardinality. Then there is a matching*

$$M : S - X \to X - S.$$

The following lemma is a generalization of Berge's Lemma.

Lemma 1.2 ([9, Lemma 3]). *Let Γ be a non-empty collection of maximum independent sets and let S be an independent set. Then there is a matching*

$$M : S - \bigcap \Gamma \to \bigcup \Gamma - S.$$

Lemma 1.3 ([8, Corollary 2.7]). *Let Γ be a non-empty collection of maximum independent sets and let Γ' be a non-empty subcollection of Γ. Then*

$$|\bigcap \Gamma'| + |\bigcup \Gamma'| \leq |\bigcap \Gamma| + |\bigcup \Gamma|.$$

In the current paper, we generalize Lemmas 1.1, 1.2 and 1.3. We study three versions of Berge systems (and three parallel versions of multi-matching systems):

(1) the general version: a pair (S, F) is a Berge system,
(2) F is a Berge system for maximum (the main version in Section 2) and
(3) a Berge system with maximality (the main version in Sections 4 and 5).

In Section 2, we define 'a graph ideal'. Theorem 2.7 proves that every graph ideal is a Berge system for maximum (a generalization of Berge's Lemma, Fact 1.1). Theorem 2.14 characterizes the graph ideals that are closed under isomorphisms. Theorem 3.16 is a generalization of Lemmas 1.2 and 1.3. In Sections 4 and 5, we study Berge systems with maximality and multi-matching systems respectively, for groups in the context of non-commuting graphs. In Section 6, we give an example of a ring, showing that our results concerning groups fail for rings.

2. Graph ideals as Berge systems

The notion of an ideal is central in several branches of mathematics. In this section, we define 'a graph ideal', which is a natural generalization of 'an ideal' in the context of graph theory. We define 'a Berge system for maximum', in order to generalize and extend Berge's Lemma. By Theorem 2.7, every graph ideal is a Berge system for maximum. Theorem 2.14 characterizes graph ideals that are closed under isomorphisms.

Definition 2.1. A *Berge system for maximum* (in $\mathbb{G}$) is a collection F of vertex sets such that for each $S, X \in F$, if $|X|$ is maximum in $\{|A| : A \in F\}$, then there is a matching from $S - X$ into X.

Definition 2.2. An ideal is a collection of sets which is closed under unions and satisfies the hereditary property.

Definition 2.3. Let $\mathbb{G}$ be a graph. A collection of vertex subsets F is a *graph ideal* for $\mathbb{G}$ if:

(1) (hereditary property) $A \subset B$ and $B \in F$ imply $A \in F$,
(2) (closed under a stronger "restricted" union) $A, B \in F$ with $(A - B, B - A) = \emptyset$ imply $A \cup B \in F$.

The second condition may be interpreted as a closure under unions, which does not add edges.

Example 2.4. Clearly, for every graph $\mathbb{G}$, the power set of $V(\mathbb{G})$ is a graph ideal. Moreover, Every ideal of vertex subsets is a graph ideal.

Example 2.5. By Berge's Lemma (Fact 1.1), for every graph $\mathbb{G}$ the collection $Ind(\mathbb{G})$ of independent sets in $\mathbb{G}$ is a Berge system for maximum. Actually, $Ind(\mathbb{G})$ is a graph ideal: it satisfies the hereditary property and if S_1 and S_2 are independent sets and $(S_1 - S_2, S_2 - S_1) = \emptyset$, then $S_1 \cup S_2$ is independent.

Example 2.6. Let H be a set of cliques. Let F be the collection of subsets, S, of $\mathbb{G}$, such that for every $A \in H$, $|S \cap A| \leq 1$. Then F is a graph ideal.

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